Strategic Exercise of European Warrants

Nikunj Kapadia[1] Gregory Willette

Original Version: March 27, 2002
Current Version: October 8, 2005

[1]Isenberg School of Management, University of Massachusetts. The paper has benefited from comments of N. Chidambaran, Hossein Kazemi, George Martin, Sanjay Nawalkha, N. R. Prabhala, Raghu Sundaram, seminar participants at Georgetown University, University of Maryland, and the University of Virgina, and the report of an anonymous referee. I thank Xiaoling Pu for computational assistance. The paper includes results that were previously circulated in a paper titled Dilution and the Valuation of Options and Non-Identical Warrants. Please address correspondence to Nikunj Kapadia, Isenberg School of Management, 121 Presidents Drive, University of Massachusetts, Amherst, MA 01003, 413-545-5643, or email: nkapadia@som.umass.edu.

Abstract

Strategic Exercise of European Warrants

We develop a European warrant pricing theory that accounts for other securities in the capital structure of the firm, besides the stock and a warrant. We demonstrate that an individual who owns a European warrant strategically determines his exercise strategy taking into account any transfer of wealth from stockholders to holders of other non-expiring securities. In equilibrium, a warrant-holder reduces the quantity exercised to mitigate any loss of value from existing and new shareholders. However, the extent to which the warrant-holder is successful in doing so is impacted by the concentration of warrant-holding. In the extreme case, in a perfectly competitive equilibrium, competition between warrant-holders reduces the payoff on exercise to zero for a range of firm values.

The important difference between the application of the contingent claim theory of Black and Scholes (1973) and Merton (1973) to the pricing of options versus its application to the pricing of warrants is that the exercise of the latter changes the capital structure of the firm. Not surprisingly, there has been much written on the pricing of warrants and related securities,[1] and an extensive literature focuses on understanding the economic consequences of a warrant-holder's exercise policy.

In particular, it has been noted that strategic and competitive interaction between warrant- and stock-holders determines the early exercise policy and value of the American warrant. Both exercise policy and warrant price depend on the concentration of warrant-holding - how widely warrants are distributed across individual warrant-holders. The exercise policy when warrants are owned by a monopolist (Emanuel (1983)) is not the same as when warrants are held by several individual warrant-holders (eg. Constantinides (1984), Spatt and Sterbenz (1988)). Ingersoll (1987) provides a lucid review of this literature. A related and important literature has investigated exercise policies for holders of real options (see, in particular, Williams (1993) and Grenadier (1996, 2002)).

However, there is little or no discussion of competitive or strategic considerations for the exercise of the European warrant. In fact, Galai and Schneller (1978) demonstrate that the European warrant is equivalent to a diluted call on the value of the firm. Both warrant and call are exercised fully when they are in the money and expire worthless otherwise. Curiously, the exercise strategy of the call-holder whose actions have no impact on the stock price is identical to that of a warrant-holder whose exercise impacts the stock price. We motivate this paper by demonstrating that the above case is unusual in that the strategic response of the warrant-holder coincides with that of a non-strategic investor, and this is so because of the commonly made assumption that there are no other securities, besides a warrant and stock, in the capital structure of the firm. When there are other securities in the capital structure of the firm, it is no longer true that these exercise strategies coincide. In fact, competitive and strategic considerations matter, and the warrant-holder's exercise decision is influenced by all the securities in the capital structure as well as by the concentration of warrant-holding.

Our results rest on a simple insight. By changing the capital structure of the firm, the exercise of a warrant potentially affects *all* risky securities in the capital structure. However, if exercise of a warrant affects these other securities, then the presence of other securities will, in turn, impact the equilibrium exercise strategy. Thus, any analysis of exercise strategies

[1] The literature includes Ingersoll (1977) and Brennan and Schwartz (1977) for convertible bonds, Galai and Schneller (1978) for the European warrant, and Emanuel (1983), Constantinides (1984), Constantinides and Rosenthal (1984) and Spatt and Sterbenz (1988) for American warrants.

that does not explicitly account for the rest of the capital structure of the firm, besides the warrant and stock, is incomplete. Once we allow for the presence of other securities, we have to explicitly model - and solve - the warrant-holder's optimization problem. Strategic and competitive considerations then play a role in determining the equilibrium exercise strategy of the European warrant-holder.

In exercising the European warrant at maturity, a warrant-holder makes two related decisions. One, the warrant-holder decides the exercise threshold at which to start exercising the warrant. Two, he decides the number of warrants to exercise, knowing full well that any unexercised warrants will expire worthless. Both these decisions can be jointly formulated as being that of determining the warrant-holder's optimal exercise policy as a function of the underlying value of the firm, where the exercise policy is defined as the fraction of warrant-holding that is exercised at maturity. The optimal policy is chosen to maximize the value of the individual's own warrant-holding, after accounting for the effect of exercise on the stock and securities held by other individuals as well as the exercise policies of other individuals holding the warrant.

We demonstrate that the presence of longer maturity securities - upstream securities that mature after the warrant - plays a critical role in determining how many warrants are exercised. In order to maximize the total value of his warrants, an individual warrant-holder may *partially* exercise his warrant-holding, and let the remaining warrants expire worthless. The partial exercise strategy is analogous to the sequential exercise of American warrants as first noted by Emanuel (1983).

Why is partial exercise an equilibrium strategy? Given a dividend policy, exercise of warrants may change the scale of the firm and the number of outstanding shares. The net impact might be that upstream securities (as, for example, other longer-dated warrants) gain in value. The increase in value of these securities negatively impacts the stock price, hurting both existing shareholders and exercising warrant-holders. The equilibrium response of a warrant-holder who does not benefit from the gain in value of upstream securities is to reduce the quantity exercised to mitigate the negative impact of his exercise, and this may include exercising only a fraction of his holdings.

How successfully warrant-holders can reduce wealth transfer to upstream securities depends on the concentration of warrant-holding. When there is greater competition between individual warrant-holders, we expect warrant-holders to be less effective in mitigating the negative impact of exercise. In fact, in a competitive equilibrium, we demonstrate that over the range of firm values for which warrants are partially exercised, competition between exercising warrant-holders reduces the payoff to the exercised warrant to zero. Any increase in firm value in this range solely benefits upstream securities, while the stock price remains fixed at the strike of the warrant, and the value of both exercised and unexercised warrants is zero.

The perfectly competitive equilibrium is of both theoretical and practical importance as it has been noted that warrants are often widely held across individuals (Spatt and Sterbenz (1988)). We demonstrate how to compute the number of warrants partially exercised in the perfectly competitive equilibrium. This is particularly important as the exercise strategy must be determined before one can value a security. This contrasts the conclusion in Constantinides (1984) in whose model there are no additional securities in the capital structure, and the American warrant price can be derived without explicit computation of the exercise strategy. Our analysis, in fact, suggests that when there is a more complex capital structure, the exercise strategy has to be determined for valuing all securities. In addition to demonstrating how the exercise strategy can be computed, we provide guidance on how it may be used consistently for valuation, and also provide a generalization of the European warrant formula of Galai and Schneller (1978).[2]

Although strategic and competitive considerations have been previously noted in the literature on the American warrant, our results have a different flavor from these prior results. Partial (sequential) exercise of American warrants is optimal as it transfers wealth from the shareholder to the exercising warrant-holder (Emanuel (1983)). Indeed, this observation has led to a discussion in the literature of how shareholders can prevent expropriation by a monopolizing warrant-holder by changing investment or dividend policies (Spatt and Sterbenz (1988)). In contrast, we point out that when individuals holding different securities in the capital structure do not collude, shareholders benefit by allowing warrants to be monopolistically held on the exercise date.

The remainder of the paper is as follows. Section 1 motivates the problem using a single period example. Section 2 describes the setup of the model. Section 3 discusses the equilibrium exercise policy in the presence of longer-maturity securities. Section 4 applies the model to valuation, and Section 5 discusses some implications for the empirical literature. The last section concludes.

[2]Darsinos and Satchell (2002) and Dennis and Rendleman (2003) consider valuation of multiple tranches of warrants. Neither of these papers model or consider the warrant-holder's equilibrium exercise policy. In a previous version of this paper, we reported sample values of multiple tranches of warrants; these are available on request.

1 A Motivating Example

A one-period example, following Ingersoll (1987), suffices to demonstrate the importance of considering the entire capital structure of the firm for determining the exercise strategy and value of a warrant. Consider a firm with a market value of assets of $V(t)$ at time t. Over the next period to time T, the market value may either double or halve with risk-neutral probabilities of 1/3 and 2/3, respectively. The risk-free rate is zero, the firm does not pay dividends, and cash inflow from exercise of warrants is re-invested to increase the scale of the firm. The firm is financed by a set of securities, each of which are held by a different set of individuals.

First, consider the traditional case analyzed in the literature where the capital structure of the firm consists of only common stock and a single warrant series. Assume that the firm has 100 shares of common stock S, and 50 warrants W of strike 90 that expire at t. Let the current firm value be $V(t) = \$10,000$. Given the current firm value, if all 50 warrants are exercised, then the cash inflow of 50 x 90 will make the firm worth \$14,500, and the number of shares will increase to 150. The value of each share will be $\$14,500/150 = \$96.67$. The payoff for each warrant is (96.67-90)=\$6.67, and all 50 warrants will get exercised. In fact, it can be verified that it is optimal to exercise all 50 warrants for every firm value greater than \$9,000. For firm values less than \$9,000, the warrants will expire worthless. In other words, as is well known, the exercise decision of the holder of a European warrant, analogous to that of an individual who owns a call option, is to exercise all warrants if the underlying firm value per share is greater than the strike, and to let warrants expire worthless otherwise.

Next, consider how the exercise decision is affected when the capital structure of the firm includes an additional 50 warrants, denoted by $\bar{W}$, with strike 112 and expiration date of T. The value of $\bar{W}$ will depend on whether or not W is exercised at t, so we can write the warrant price at t as $\bar{W}(V, m)$, where m is the number of expiring warrants W that are exercised, $0 \leq m \leq 50$.

Suppose we assume that all expiring warrants are exercised, $m = 50$. Then, as already noted, the current value of assets increases to \$14,500 and the number of shares to 150. At T, the warrant $\bar{W}$ will be exercised if the firm value doubles to \$29,000. After exercise of these additional 50 warrants of strike 100, the stock price at T will be $(29,000 + 50 \text{ x } 112)/(150 + 50) = 173$. If the firm value halves, $\bar{W}$ will not be exercised and the stock price will be $7,250/150 = 48.33$. From $\bar{W}(T)$, $S(T)$, and the risk-neutral probabilities, we can compute prices of $\bar{W}(t)$ and $S(t)$. At t, the non-expiring warrant is worth

$$\bar{W}(V = 10,000, m = 50) = \frac{1}{3}(173 - 112) = \$20.33.$$

and the stock price is,

$$S(V = 10,000, m = 50) = \frac{1}{3}173 + \frac{2}{3}48.33 = \$88.89.$$

We can now verify whether our assumption that $m = 50$ is correct. In fact, $m = 50$ is not an equilibrium exercise strategy as W is out-of-the-money at the stock price of 88.89.

So then should W not be exercised? To check this, we again value both $\bar{W}$ and S with $m = 0$. At T_1, the firm value either doubles to \$20,000 or halves to \$5,000. $\bar{W}$ will be exercised if the firm value is 20,000, and the stock price at T_1 will then be $(20,000 + 50$ x $112)/(150) = \$170.67$. If the firm value halves, the warrant $\bar{W}$ will not be exercised and the stock price will be equal to \$50. The time t values of $\bar{W}$ and S are then,

$$\bar{W}(V = 10,000, m = 0) = \frac{1}{3}(170.67 - 112) = 19.56,$$

and

$$S(V = 10,000, m = 0) = \frac{1}{3}170.67 + \frac{2}{3}50.00 = 90.22.$$

The stock price of 90.22 is *higher* than the expiring warrant's strike of 90. Therefore, $m = 0$ is also not an equilibrium strategy.

As we have ruled out both $m = 50$ and $m = 0$, it must be that the optimal exercise strategy is one where warrant-holders exercise some of the warrants, and let the remainder expire worthless. The example illustrates that it may be important for the warrant-holder to strategically determine his exercise policy after taking into account the impact of exercise on the other securities in the capital structure. Once we allow for strategic exercise, there are three novel implications for warrant valuation that render existing European warrant pricing theory inadequate.

First, as already noted, the space of exercise policies is not limited to exercising all or none of the warrants. Instead, for each firm value, the warrant-holders have to decide how many warrants to exercise. Second, we cannot determine the precise number of warrants that should be exercised without modeling the level of competition between exercising warrant-holders. In the above example, although the total payoff of $m(S(V, m) - X)$ is maximized if only 11 warrants are exercised, upto 25 of the 50 warrants can be exercised profitably, Unless a single warrant-holder monopolistically holds all outstanding warrants, it is not likely that 11 warrants are exercised in equilibrium. This then brings us to the final implication. The value of any of the securities in the capital structure - W, $\bar{W}$ or S - cannot be determined without first determining the number of warrants that are exercised in equilibrium. In other words, a central problem of valuation of securities in the presence of warrants (or, more generally, any dilutive security) is the determination of the equilibrium exercise policy of the warrant-holder.

2 The Model

2.1 Setup

Consider a non-dividend paying firm with 1 share and firm value per share of $V(t) > 0$ at time $t \equiv T_0$ with European warrants of maturity $T_1 \geq t$ issued on its common stock. Let the warrant of strike X be denoted as W. The number of outstanding warrant is $\alpha \geq 0$. Holders of the warrants have a right to purchase and create one share in the issuing firm. As we have normalized the number of shares to 1, we can interpret α as the dilution factor of the warrant - the additional number of shares created by exercise of warrants for every original share. In addition to warrants of maturity T_1, the firm may have other outstanding securities (including warrants) of maturity greater than T_1. We designate the total time t value of these securities as $\bar{W}(t)$. Although the firm does not pay regular dividends, it may choose to pay an extraordinary dividend from the proceeds of warrant exercise.

By the absence of arbitrage, for all $t \leq T_1$,

$$V(t) = S(t) + \alpha W(t) + \bar{W}(t), \tag{1}$$

where S is the stock price. As we focus on exercise strategies of expiring warrants at T_1 and because we do not model bankruptcy prior to this date, we do not model securities in the capital structure that mature on or before T_1. Equivalently, we may consider V to be the firm value process less the value of any such securities. (We can relax the assumption that the warrants maturing at T_1 have the same strike.) For the illustrations in the paper, we will assume that $V(t)$ is lognormally distributed under the risk-neutral measure Q,

$$dV(t) = rV(t)dt + \sigma V(t)\, dW^Q(t), \tag{2}$$

where r is the risk-free rate, σ is the volatility, and W^Q is the Wiener process.

Warrants may be held by a single or several individuals, each of whom optimally exercise their holding. Specifically, we assume that the warrant is held by a constant $K \geq 1$ individuals, indexed by $k = 1, 2, ..., K$, where each holder of the warrant holds none of the other securities. When $K = 1$, a single monopolist holds all of the outstanding warrant (Emanuel (1983)). Our focus, however, will be the case of the perfectly competitive equilibrium, $K \to \infty$ (Constantinides (1984), Constantinides and Rosenthal (1984)). Denote α^k as the amount held by investor k so that $\alpha = \sum_k \alpha^k$. We define each investor's *exercise policy*, $\delta^k(V(T_1))$, as the fraction of α^k exercised at T_1, $0 \leq \delta^k \leq 1$. The individual's equilibrium strategy is δ^{k*}. When $K > 1$, we assume a Nash equilibrium where each investor optimally exercises δ^{k*} of α^k conditioned on other warrant holders exercising an optimal amount of their own holdings. Define $\delta(V(T_1); K)$ as the fraction of outstanding warrants exercised at equilibrium, $\delta = (\sum_k \delta^{k*}\alpha^k)/(\sum_k \alpha^k)$. We expect δ to be non-decreasing in $V(T_1)$.

Define $\bar{\alpha}(t) \equiv \bar{\alpha}(V(t))$, $t \leq T_1$, as the number of new shares created by exercise of warrants, $\bar{\alpha}(t) = \delta\alpha$, if $t = T_1$ and zero otherwise. Prices of all securities in the capital structure will depend on $V(t)$ and $\bar{\alpha}(t)$: $W(t) \equiv W(V(t), \bar{\alpha}(t), t)$, $S(t) \equiv S(V(t), \bar{\alpha}(t), t)$ and $\bar{W}(t) \equiv \bar{W}(V(t), \bar{\alpha}(t), t)$. We assume that W, S, $\bar{W}$ are twice-differentiable in their first argument, and at least once-differentiable in their second and third arguments (every exercise strategy δ^k). This can be verified once $\bar{W}$ is parameterized. To focus on the effect of exercise by a warrant-holder, we will suppress other dependencies and write $S \equiv S(V, \delta^k(V))$, $\bar{W} \equiv \bar{W}(V, \delta^k(V))$.

We assume that $0 < \partial\bar{W}/\partial V < 1$. The assumption that $\partial\bar{W}/\partial V \neq 0$ for all $V > 0$ implies that $\bar{W}$ is risky. This assumption eliminates the trivial case when $\bar{W}$ is riskless debt, and, for most common securities, ensures that exercise of warrants affects the value of $\bar{W}$. To understand the usefulness of the upper bound on $\partial\bar{W}/\partial V$, observe that, for a given $S(T_1)$ and unobserved $V(T_1)$, equation (1) is a fixed-point problem. If $\partial\bar{W}/\partial V < 1$, then for any observed stock price (and δ), there is a unique V by the contraction mapping theorem. This assumption is satisfied by many common securities in the capital structure. It can be easily verified to be true when $\bar{W}$ is a warrant of maturity $T_2 > T_1$. It is also satisfied if $\bar{W}$ is risky debt as in Merton (1974).[3]

[3]The assumption imposes restrictions on the stochastic process followed by V. For instance, when $\bar{W}$ is a warrant of maturity $T_2 > T_1$, the value of $\bar{W}$ on expiration of W is equivalent to that of a diluted call written on V (after suitable normalization for any extra shares created on exercise). Then to ensure that $\partial\bar{W}/\partial V < 1$, we must have the delta of the call option be bounded by 1. Bergman, Grundy and Wiener (1996) provide the restrictions that must be imposed on V for this to hold. For most standard models for $V(t)$ (lognormal, Heston's (1993) stochastic volatility model), the delta of the call is, in fact, bounded by 1.

The payoff of the warrant, W, on exercise is

$$W(T_1) = \max(S(T_1) - X, 0). \tag{3}$$

Non-exercised warrants expire worthless. The functional dependence of W on $V(T_1)$ can be made explicit by substituting for $S(T_1)$ using equation (1).

On exercise of warrants at T_1, the number of outstanding shares increases from 1 to $1 + \sum_{k=1}^{K} \delta^k \alpha^k$, and the total value of the firm can increase upto $\sum_{k=1}^{K} \delta^k \alpha^k X$ with the cash inflow from exercise, less any extraordinary dividend that the firm chooses to pay from the proceeds of exercise. To maintain the convention of normalizing the number of shares to 1 and using the superscript $+$ to denote post-exercise values, let $V^+(T_1)$ be the value per share of the firm after exercise, and $D(V, \delta)$ be the extraordinary dividend per share. The relation between $V(T_1)$ (before exercise) and $V^+(T_1)$ (after exercise) is,

$$V^+(T_1) = \frac{V(T_1) + \sum_{k=1}^{K} \delta^k \alpha^k X - D}{1 + \sum_{k=1}^{K} \delta^k \alpha^k}. \tag{4}$$

If the value of $\bar{W}$ is affected by the change in firm value or the number of outstanding shares, then the warrant-holder will take this into account in exercising his warrants. The motivation of this paper is to understand warrant exercise when, indeed, $\partial \bar{W} / \partial \delta^k \neq 0$.

2.2 Equilibrium Exercise Strategies

In solving for the equilibrium exercise strategy, it will be convenient to define the following two sets of exercise strategies. First, define the standard *block exercise* strategy that is analogous to the exercise strategy of an individual holding a call option.

Definition 1 *Block Exercise Strategy: The equilibrium exercise strategy δ^k is a block strategy if there exists a threshold $\bar{V}$ such that,*

$$\begin{aligned} \delta^k &= 1 \qquad \textit{if } V(T_1) > \bar{V} \\ &= 0 \qquad \textit{if } V(T_1) < \bar{V} \end{aligned}$$

The exercise policy of a call-holder is a block-exercise with the exercise threshold equal to the strike of the call.

In the context of early-exercise of American warrants, both Constantinides (1984) and Emanuel (1983) note that warrants may be exercised partly over time. Analogously, we define a *partial exercise* strategy for European warrants as follows:

Definition 2 *Partial Exercise Strategy: The equilibrium exercise strategy, δ^k, is a partial exercise strategy if, for $V^d \leq V^u$ and $0 \leq \hat{\delta}^k \leq 1$,*

$$\begin{aligned} \delta^k &= 1 \qquad \textit{if } V(T_1) > \bar{V}^u \\ &= \hat{\delta}^k(V(T_1)) \qquad \textit{if } \bar{V}^d \leq V(T_1) \leq \bar{V}^u \\ &= 0 \qquad \textit{if } V(T_1) < \bar{V}^d \end{aligned}$$

where $\hat{\delta}^k(V)$ is non-decreasing in V. The definition of the partial exercise strategy includes the block exercise as a special case when $V^d = V^u$. However, the case of interest is when $V^d < V^u$, and there is a non-trivial exercise strategy where a fraction of outstanding warrant are exercised and the remainder expire worthless.

2.3 First Order Condition for Partial Exercise

At T_1, the k'th investor who holds α^k of warrant W chooses a exercise policy to maximize the value of his holding, $\delta^k\alpha^k W(T_1)$, given that other individuals also optimally exercise their holding, and subject to the constraint that $0 \leq \delta^k \leq 1$. We will defer the discussion of when this constraint is binding (i.e., either all or none of the warrant-holding is exercised) to the next section, and focus first on partial exercise.

The first-order condition for partial exercise can be derived by substituting $W(T_1) = S(T_1) - X$ into the constraint imposed by (1) at T_1, $V(T_1) = S(T_1) + \sum_{k' \neq k} \delta^{k'*}\alpha^{k'}(S(T_1) - X) + \delta^k\alpha^k W(T_1) + \bar{W}(T_1)$. Observe that maximizing $\delta^k\alpha^k W(T_1)$ subject to (1) is equivalent to maximizing $V(T_1) - S(T_1) - \sum_{k' \neq k} \delta^{k'*}\alpha^{k'}(S(T_1) - X) - \bar{W}(T_1)$. Differentiating the latter expression with respect to δ^k, we get the first-order condition (FOC) as,

$$\frac{\partial S(T_1)}{\partial \delta^k}\left(1 + \sum_{k' \neq k} \delta^{k'*}\alpha^{k'}\right) + \frac{\partial \bar{W}(T_1)}{\partial \delta^k} = 0. \tag{5}$$

Equation (5) states that the warrant-holder makes his optimal decision by accounting for the price impact of exercise on the other securities in the capital structure.

Moreover, as $V(T_1) - S(T_1) - \sum_{k' \neq k} \delta^{k'*}\alpha^{k'}(S(T_1) - X) - \bar{W}(T_1) = \delta^k\alpha^k(S(T_1) - X)$, we can also write (5) as $\frac{\partial\left(\delta^k\alpha^k(S(T_1)-X)\right)}{\partial\delta^k}=0$. That is, the FOC also implies,

$$\alpha^k(S(T_1)-X)+\delta^k\alpha^k\frac{\partial S(T_1)}{\partial\delta^k}=0. \tag{6}$$

From (3), the warrant is exercised only if $S(T_1)-X\geq 0$, so that when equation (6) is satisfied, $\partial S(T_1)/\delta^k\leq 0$. The exercising warrant-holder trades off the gains from exercise (first term) against the change in the stock price that occurs on exercise (second term). A non-strategic warrant-holder, ignoring the price impact of his (and other individuals') exercise, will omit the second term in making his decision, exercise the warrant if it is in-the-money, and let it expire worthless otherwise. If $\bar{W}=0$, then equations (5) and (6) are satisfied only when $\frac{\partial S(T_1)}{\partial\delta^k}=0$ and $S(T_1)=X$.

2.4 Strategic and Non-Strategic Exercise

In existing literature, the warrant pricing problem has been solved for the special case when the capital structure consists of only the stock and a single warrant series, and $\bar{W}\equiv 0$. Let $C(t)\equiv C(V(t),X;T-t)$ be the value of a call of strike X and maturity $T-t$. Then,

$$W(t)=\frac{1}{1+\alpha}C(V(t),X;T-t). \tag{7}$$

An implication of this equivalence between the warrant and the call option is that the exercise policy of a warrant-holder is precisely the same as that of the call-holder. The following lemma explains why this is the case.

Lemma 1 *If $\bar{W}\equiv 0$, the following statements are equivalent,*

i. $\frac{\partial S(T_1)}{\partial\delta^k}\lesseqqgtr 0$

ii. $S(T_1)\gtreqqless X$

i.e., the optimization problem solved by a non-strategic investor who ignores the price impact of exercise is isomorphic to that of the strategic investor who does not ignore the price impact.

The equivalence of (i) and (ii) can be shown as follows. From (1) and (3), the stock price after exercise at T_1 is,

$$S(T_1) = \left(V(T_1) + \sum_{k' \neq k} \delta^{k'*} \alpha^{k'} X + \delta^k \alpha^k X \right) / \left(1 + \sum_{k' \neq k} \delta^{k'*} \alpha^{k'} + \delta^k \alpha^k \right),$$

and, differentiating and simplifying using (1), we get

$$\frac{\partial S(T_1)}{\partial \delta^k} = \frac{-\alpha^k}{(1 + \sum_{k' \neq k} \delta^{k'*} \alpha^{k'} + \delta^k \alpha^k)} \left(S(T_1) - X \right), \tag{8}$$

from which it immediately follows that $(\partial S(T_1)/\partial \delta_1^k \lesseqgtr 0) \Leftrightarrow (S(T_1) - X_1 \gtreqless 0)$.

Lemma 1 explains why, when $\bar{W} \equiv 0$, the exercise policy of a strategic investor coincides with that of the non-strategic investor. From the first-order condition of equation (5), the strategic investor takes into account the impact of his exercise policy on the stock price. In contrast, a non-strategic individual ignores this price impact and instead follows a simpler rule of exercising all warrants when $S(T_1) > X$, and none when $S(T_1) < X$. Lemma 1 demonstrates that these two conditions are isomorphic, i.e., whenever $S(T_1) > X$ ($S(T_1) < X$), it is also true that $\frac{\partial S(T_1)}{\partial \delta^k} < 0$ ($\frac{\partial S(T_1)}{\partial \delta^k} > 0$). Moreover, $S(T_1) = X$ if and only if $\frac{\partial S(T_1)}{\partial \delta^k} = 0$. In other words, when $\bar{W} \equiv 0$, the exercise policy for the non-strategic investor coincides with that of the strategic investor. Consequently, we can compute the correct equilibrium policy without explicitly evaluating (5) and considering the price impact of exercise.

3 Equilibrium Exercise Polices when $\bar{W} \neq 0$

3.1 Exercise Policy when $\partial \bar{W}/\partial \delta^k < 0$

When there are longer-dated securities in the capital structure of the firm that are impacted by warrant exercise, the warrant-holder will account for the change in the value of $\bar{W}$ in determining his exercise policy (as noted in equation (5)). The optimal exercise policy depends critically on whether exercise increases or decreases the value of $\bar{W}$, i.e., the sign of $\partial \bar{W}/\partial \delta^k$. The sign of $\partial \bar{W}/\partial \delta^k$ will, in general, depend on $V(T_1)$.

The following lemma establishes the exercise strategy for firm values where exercise decreases the value of $\bar{W}$.

Lemma 2 *If $\partial \bar{W}/\partial \delta^k < 0$ at $V(T_1) = \bar{V}$ and for all $\delta^k \in [0, 1]$, then $\delta^{k*}(\bar{V}) \in \{0, 1\}$.*

If exercise of W decreases the value of $\bar{W}$, then partial exercise cannot be an equilibrium strategy. It is useful to explicitly consider why the above statement is true. We noted that the necessary conditions for partial exercise are equations (5) and (6). If both of these are not simultaneously satisfied, then we can rule out $\delta^{k*} \in (0, 1)$. Equation (6) implies that $\alpha^k(S - X) + \delta^k \alpha^k \frac{\partial S}{\partial \delta^k} = 0$ with $S - X \geq 0$ and $\partial S/\partial \delta^k \leq 0$. Thus, equations (5) and (6) can be simultaneously satisfied only if $\partial \bar{W}/\partial \delta^k \geq 0$. It follows that when $\partial \bar{W}/\partial \delta^k < 0$, we get a corner solution, $\delta^{k*} \in \{0, 1\}$. Intuitively, when exercise of the warrant decreases the value of upstream securities $\bar{W}$, there is a wealth transfer from $\bar{W}$ to the stock-holders and the exercising warrant-holders, and the stock price increases as more warrants are exercised. Thus, exercising all of an at- or in-the-money warrant gives the highest payoff to an individual warrant-holder.

The above reasoning also indicates why we need to consider strategies that are not in $\{0, 1\}$. For most common securities in the capital structure, including when $\bar{W}$ is another warrant or risky debt, $\partial \bar{W}/\partial \delta^k$ may be strictly positive over some range of firm values.

We begin by providing a detailed illustration to demonstrate that, in fact, a partial exercise policy is an equilibrium policy, and that the number of warrants partially exercised depends on K. (An extension, in a later section, will also help illustrate Lemma 2 where there is no partial exercise in equilibrium.) In the illustration, we assume $\bar{W}$ is a second warrant of maturity greater than T_1. We specifically choose this illustration of two tranches of warrants as it demonstrates a wide range of exercise strategies that are possible in equilibrium.[4]

3.2 An Illustration: Strategic Equilibrium with Partial Exercise

Let $\bar{W}$ be $\bar{\alpha}$ upstream warrants of maturity $T_2 > T_1$ and strike $\bar{X}$. On exercise of W at T_1, from (7), the value of $\bar{W}$ is,

$$\bar{W}(T_1) = \frac{\bar{\alpha}}{1 + \bar{\alpha}^+} C(V^+(T_1), \bar{X}; T_2 - T_1), \tag{9}$$

where

$$\bar{\alpha}^+ = \frac{\bar{\alpha}}{(1 + \sum_{k'=1}^{K} \delta^{k'} \alpha^{k'})} \tag{10}$$

[4]Two or more tranches of warrants are also common across international markets. In Schulz and Trautmann's (1994) empirical investigation of German warrants, more than half the firms in their sample have multiple series of warrants. In Japan, over the last decade, many companies have had multiple series of warrants or convertible bonds outstanding. For instance, in 1996, these firms included Daiwa Securities with 10, Nomura Securities with 7, Matsushita Electric with 4, and Sony Corporation with 5 issues (BZW Japanese Company Derivative Handbook, 1996.)

is the dilution factor for $\bar{W}$ adjusted for the new shares created at T_1 on exercise of W. In this section, we will assume that the firm does not pay any extraordinary dividend from the proceeds of exercise so that, from (4) with $D = 0$, $V^+(T_1) = (V(T_1)+\sum_{k'=1}^{K} \delta^{k'}\alpha^{k'} X)/(1+\sum_{k'=1}^{K} \delta^{k'}\alpha^{k'})$. (We will also, in a later section, separately consider the case when $D > 0$.)

We will use equation (6) to understand the partial equilibrium strategy. First, evaluate $\partial S(T_1)/\partial\delta^k$ by noting that conditioned on the k'th warrant-holder exercising δ^k, $V(T_1) = S(T_1) + \delta^k\alpha^k(S(T_1) - X) + \sum_{k'\neq k} \delta^{k'}\alpha^{k'}(S(T_1) - X) + \bar{W}(T_1)$. Differentiating this expression with respect to δ^k,

$$\frac{\partial S(T_1)}{\partial\delta^k} = \frac{-\alpha^k(S(T_1) - X)}{1 + \sum_{k'=1}^{K} \delta^{k'}\alpha^{k'}} - \frac{1}{1 + \sum_{k'=1}^{K} \delta^{k'}\alpha^{k'}} \frac{\partial\bar{W}(T_1)}{\partial\delta^k}. \tag{11}$$

In addition, by differentiating equation (9) we get,

$$\frac{\partial\bar{W}(T_1)}{\partial\delta^k} = \left(\frac{\bar{\alpha}^+}{1+\bar{\alpha}^+}\right)\left(\frac{\alpha^k}{1 + \sum_{k'=1}^{K} \delta^{k'}\alpha^{k'}}\right)\left[\bar{W}(T_1) - \Delta(T_1)\left(V(T_1) - X_1\right)\right], \tag{12}$$

where $\Delta(T_1) = \partial C(V^+(T_1), \bar{X}; T_2 - T_1)/\partial V^+(T_1)$ is the "delta" of the call. Note that $\partial\bar{W}(T_1)/\partial\delta^k$ is negative when W is deep in the money ($V(T_1) >> X$), but positive when W is at- or close- to the money. From Lemma 2, if there is partial exercise of warrants in equilibrium, it will be for firm values where $\partial\bar{W}(T)/\partial\delta^k > 0$. Not surprisingly, these are firm values where the warrant is close-to-the-money.

Substituting (11) and (12) into equation (6) and simplifying, we get,

$$\frac{\partial\left(\delta^k\alpha^k W(T_1)\right)}{\partial\delta^k} = a_1(\delta^k; K)\left(V(T_1) - X\right) - a_2(\delta^k; K)\,\bar{W}(T_1), \tag{13}$$

where

$$\begin{aligned} a_1(\delta^k; K) &= \left(\frac{\alpha^k}{1 + \sum_{k'=1}^{K} \delta^{k'}\alpha^{k'}}\right)\left[1 - \frac{\delta^k\alpha^k}{1 + \sum_{k'=1}^{K} \delta^{k'}\alpha^{k'}} + \left(\frac{\bar{\alpha}^+}{1+\bar{\alpha}^+}\right)\left(\frac{\delta^k\alpha^k}{1 + \sum_{k'=1}^{K} \delta^{k'}\alpha^{k'}}\right)\Delta(T_1)\right], \\ a_2(\delta^k; K) &= \left(\frac{\alpha^k}{1 + \sum_{k'=1}^{K} \delta^{k'}\alpha^{k'}}\right)\left[1 - \frac{\delta^k\alpha^k}{1 + \sum_{k'=1}^{K} \delta^{k'}\alpha^{k'}} + \left(\frac{\bar{\alpha}^+}{1+\bar{\alpha}^+}\right)\left(\frac{\delta^k\alpha^k}{1 + \sum_{k'=1}^{K} \delta^{k'}\alpha_1^{k'}}\right)\right]. \end{aligned}$$

If the inequality constraints are not binding and the optimal exercise strategy for the k'th warrant-holder is a partial exercise strategy, then we can compute it by solving $\frac{\partial\left(\delta^k\alpha^k W(T_1)\right)}{\partial\delta^k} = 0$ using equation (13).[5]

[5] It is important to verify that a solution of $\frac{\partial\left(\delta^k\alpha^k W(T_1)\right)}{\partial\delta^k} = 0$ is indeed an equilibrium policy (i.e., to compare with the payoff for $\delta^{k*} \in \{0, 1\}$). For the special case of a competitive equilibrium, Proposition 1 provides a

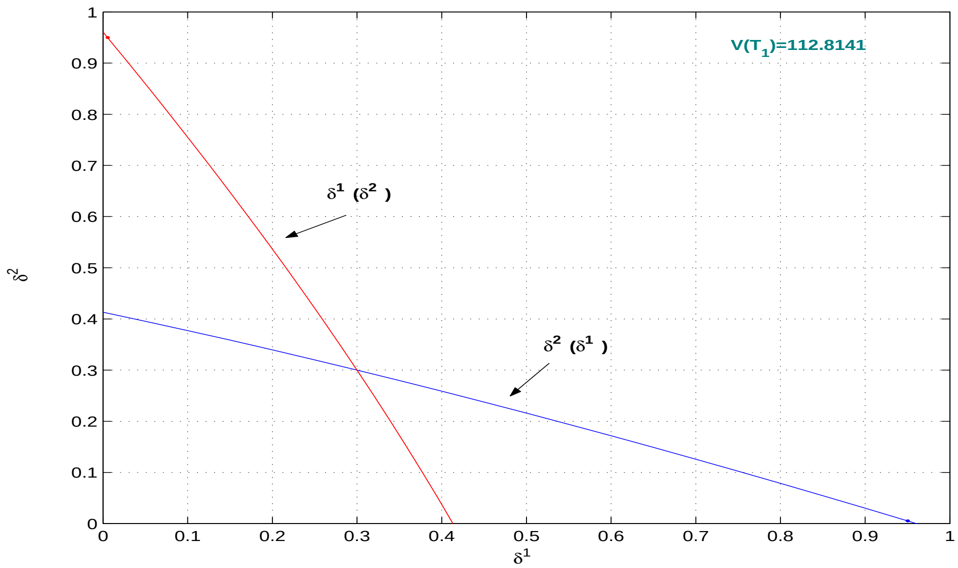

Figure 1: The graph plots the optimal exercise policy at maturity T_1 of one individual warrant-holder as a function of the the other warrant-holder's exercise policy, when two individuals equally hold the expiring warrant W, i.e., $K = 2$. There are $\bar{\alpha}$ units of an upstream warrant $\bar{W}$ of strike $\bar{X}$ and maturity T_2. The parameters used to compute the exercise policy are: $\alpha = 1$, $\bar{\alpha} = 1$, $r = 0$, $X = 90$, $\bar{X} = 100$, $T_2 - T_1 = 5$, and $V(T_1) = 112.8141$. The equilibrium exercise policy is given by the intersection of the two graphs, $\delta^{1*} = \delta^{2*} = 0.30$.

Figure 1 plots best response correspondences when two identical individuals hold all of W, i.e., $K = 2$ and $\alpha^1 = \alpha^2$. The optimal policy response for each individual is graphed as a function of the other individual's policy. The equilibrium policy is given by the intersection of the two graphs. As both individuals are identical, the individual policies are similar. In the example graphed, the fraction of the total warrants exercised in equilibrium at a firm value of 112.81 is $\delta = 0.3$. For firm values greater (less) than 112.81, more (less) warrants will be exercised. The equilibrium exercise policy is a partial equilibrium policy: there exists a range of firm values where each individual warrant-holder exercises a fraction of his holding.

Figure 2 plots δ as a function of K for the symmetric equilibrium. As K explicitly enters in equation (13), the exercise policy depends on the distribution of warrant-holding. For every K, exercise begins at a firm value of 111.12. The rate of exercise, however, differs across K. For example, when $K = 100$, all warrants are exercised at a firm value of 113.91 (the exercise policy

simple sufficient condition for the existence of a partial equilibrium policy. In general, any of the following are possible: (i) there may not exist a partial exercise strategy in equilibrium for any $V(T_1)$; (ii) some partial exercise strategies (that satisfy the equation $\frac{\partial\left(\delta^k \alpha^k W(T_1)\right)}{\partial \delta^k} = 0$) may not be optimal as they are dominated by a policy of exercising all or none of the warrants; and (iii) a partial equilibrium strategy that is an equilibrium strategy may not be the sole equilibrium strategy, and, in addition, there may also exist a non-partial equilibrium strategy. We provide an example for each of these possibilities in Section 3.4.

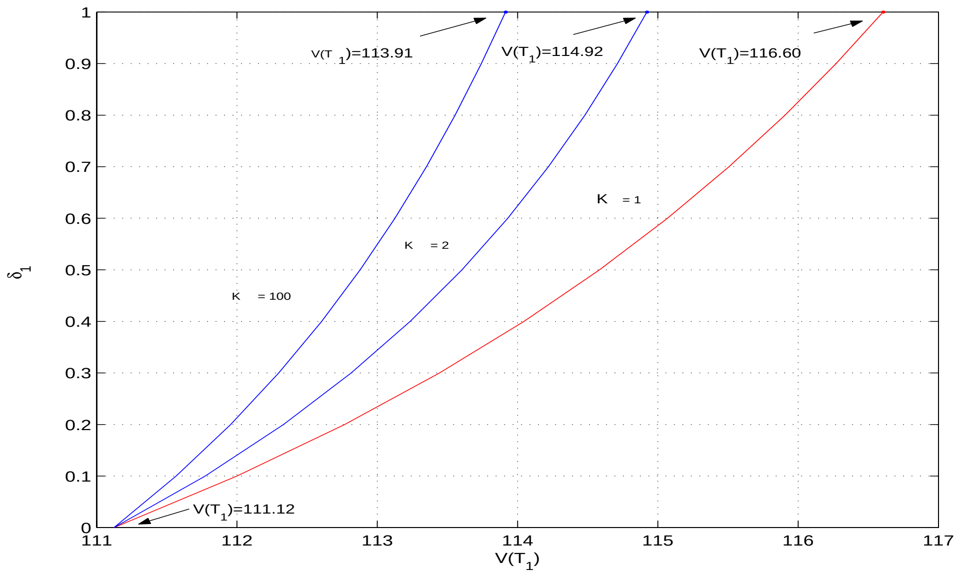

Figure 2: The graph plots the unique equilibrium exercise policy δ when K individuals hold the expiring warrant W. The equilibrium is determined by a symmetric Nash equilibrium. There are α units of an upstream warrant $\bar{W}$ of strike X and expiration date T_2. The parameters used to compute the exercise policy are: $\alpha = 1$, $\bar{\alpha} = 1$, $r = 0$, $X = 90$, $\bar{X} = 100$, $T_2 - T_1 = 5$.

for $K > 100$ is virtually identical to that for $K = 100$). On the other hand, a monopolist would exercise less than 40% of his holding at this identical firm value. Correspondingly, the value of the warrant is considerably higher when the monopolist exercises it. At $V(T_1) = 113.91$, the value of the warrant for the monopolist is \$0.71. In contrast, when $K = 100$, the value of this warrant is only about half a cent.

Why are more warrants partially exercised when K is higher? Recall from our previous discussion that when $\partial\bar{W}/\partial\delta > 0$, there is a wealth transfer from the stock-holders to the holders of the upstream securities, and the stock price is strictly decreasing in δ. The partial exercise of warrants is a response of the investor to reduce the magnitude of the wealth transfer by exercising fewer warrants. The magnitude of the wealth transfer is the least when each warrant is held by a single individual as he exercises the least of his holding. With higher K, the greater competition between individual warrant-holders leads, in equilibrium, to more warrants being exercised, a lower stock price, and lower payoff.

3.3 Competitive Equilibrium

In the illustration of the previous section, the fraction of warrants exercised in equilibrium, δ, is a solution to a rather complicated fixed point problem that depends on K. It has been noted in the literature that warrants are often held by a large number of individuals (Spatt and Sterbenz (1988)). It is then of both theoretical and empirical importance to understand the exercise policy under a perfectly competitive equilibrium. The perfectly competitive equilibrium has been previously analyzed for the case of the single American warrant by Constantinides (1984) and Constantinides and Rosenthal (1984).

In a perfectly competitive equilibrium, let the warrant-holder take the price as given, ignoring any price impact of his own exercise. From equation (6), it is clear that, for a given stock price $S(T_1)$, the equilibrium policy is to exercise all of the j'th warrant if $S(T_1) > X$, and none, if $S(T_1) < X$. Thus, the interesting case is for firm values where $S(T_1) = X$. We begin by defining this set of firm values. Let $\mathbf{V}$ be the set of firm values at T_1 defined by,

$$\mathbf{V} = \left\{ V \mid V = X + \bar{W}(V, \delta(V)),\ \delta \in [0,1] \right\}. \tag{14}$$

This set of firm values is defined by substituting $S(T) = X$ in equation (1). When $\partial\bar{W}/\partial V < 1$, the set is non-empty, and, moreover, corresponding to each $\delta \in [0,1]$, there is a unique V.

We summarize the main result in the following proposition, and then explain the result by again considering our illustrative example.

Proposition 1 *If* $\frac{\partial\bar{W}}{\partial\delta} > 0$ *for* $V \in \mathbf{V}$ *and for all* $\delta \in [0,1]$*, then there exists a perfectly competitive partial equilibrium policy defined by* $S(V, \hat{\delta}(V)) = X$*. That is,*

i. There exists a V^d *such that when* $V(T_1) < V^d$*,* $S(T_1) < X$ *for all* $\delta \in [0,1]$*;*

ii. There exists a V^u *such that when* $V(T_1) > V^u$*,* $S(T_1) > X$ *for all* $\delta \in [0,1]$*;*

iii. On the interval $\left[V^d, V^u\right]$*, there is a unique continuous increasing function* $\hat{\delta}(V)$ *defined by* $S(V, \hat{\delta}(V)) = X$*, such that*

A. $\hat{\delta}(V^d) = 0$*,*

B. $\hat{\delta}(V^u) = 1$*,*

C. for $0 \le \delta < \hat{\delta}(V)$*,* $S(V, \delta) > X$*, and*

D. for $1 \ge \delta > \hat{\delta}(V)$*,* $S(V, \delta) < X$*.*

Proof of Proposition 1: See Appendix.

Proposition 1 defines and demonstrates the existence of a stable competitive equilibrium. In addition, the proposition precisely identifies the number of warrants exercised in equilibrium. The equilibrium consists of two thresholds, V^u and V^d. When $V(T_1) \geq V^u$ ($V(T_1) \leq V^d$), δ is equal to 1 (0). When $V^d < V(T_1) < V^u$, warrants are partially exercised in equilibrium, and the precise number exercised, $\hat{\delta}(V)$, is uniquely defined by $S(V, \hat{\delta}(V)) = X$. In other words, for all $V(T_1) \in [V^d, V^u]$, just a sufficient number of warrants are exercised to make the payoff at exercise zero. In the absence of collusion, no individual warrant holder is able to profitably exercise W in the interval $[V^d, V^u]$. The stock price remains constant at X over all this range of firm values, and the value of both exercised and unexercised warrants is equivalently zero. In his analysis of the competitive equilibrium for the early exercise of an American warrant, Constantinides (1984) also demonstrates an equilibrium where the value of an exercised warrant is equal to that of an unexercised warrant.

To illustrate Proposition 1, consider the example discussed in the previous section. Write the necessary condition of equation (13), $\partial\left(\delta^k \alpha^k W(T_1)\right)/\partial\delta^k = 0$, as $V(T_1) = X + \frac{a_2}{a_1}\bar{W}(T_1)$. Assume a symmetric equilibrium, $\alpha^k = \alpha/K$. Then, taking the limit $K \to \infty$ reduces the first-order condition to,

$$V(T_1) = X + \bar{W}(T_1), \tag{15}$$

which is, of course, the set of firm values corresponding to $\mathbf{V}$. For the competitive equilibrium to exist, there must exist a non-decreasing exercise policy $\hat{\delta}(V(T_1))$ for which equation (15) is satisfied for every value of $\hat{\delta} \in [0,1]$. The computation of the competitive equilibrium policy, $\hat{\delta}(V(T_1))$, can be viewed as the inverse problem of solving for $V(T_1)$ for a given $\hat{\delta} \in [0,1]$. As there is one-to-one mapping between $\hat{\delta}$ and V, each value of $\hat{\delta} \in [0,1]$ corresponds to a firm value in $[V^d, V^u]$, where V^d and V^u are the firm values corresponding to $\hat{\delta} = 0$ and $\hat{\delta} = 1$, respectively. The unique mapping from $\hat{\delta}$ to V is assured because $\partial\bar{W}/\partial V = \frac{\bar{\alpha}}{1+\bar{\alpha}^+}\partial C/\partial V < 1$. The existence of the unique competitive equilibrium is guaranteed if, in addition, $\partial\bar{W}/\partial\hat{\delta} > 0$ for $V \in \mathbf{V}$ and for all $\delta \in [0,1]$.[6] To demonstrate this, note that we can write the inverse problem of (15) as $V(\hat{\delta}) - X = \bar{W}(V(\hat{\delta}), \hat{\delta})$. Differentiating with respect to V (and applying the Implicit Function theorem), $dV/d\hat{\delta} = (\partial\bar{W}/\partial\hat{\delta})/(1 - \partial\bar{W}/\partial V)$, and is positive if $\partial\bar{W}/\partial\hat{\delta} > 0$. An application of the Inverse Function Theorem then establishes the existence of the competitive partial exercise

[6] Write $\partial\bar{W}(T_1)/\partial\delta^k$ in equation (12) as

$$\left(\frac{\bar{\alpha}^+}{1+\bar{\alpha}^+}\right)\left(\frac{\alpha^k}{1+\sum_{k'}\delta^{k'}\alpha^{k'}}\right)\left[(1-\Delta(T_1))\bar{W}(T_1) - \Delta(T_1)(1+\sum_{k'=1}^{K}\delta^{k'}\alpha^{k'})(S(T_1)-X)\right].$$

It follows that at $S(T_1) = X$, $\partial\bar{W}(T_1)/\partial\delta^k > 0$, i.e., $\bar{W}$ is increasing in δ^k when W is exercised at firm values where it is precisely at-the-money ($V \in \mathbf{V}$).

policy $\hat{\delta}(V)$.

Proposition 1 is especially useful as it demonstrates that the competitive equilibrium policy can be computed in a straightforward manner. This is particularly fortunate as the assumption that warrants are widely held is likely to be descriptive of actual warrant holding as noted by Spatt and Sterbenz (1988). For $K < \infty$, the equilibrium exercise policy must simultaneously satisfy each individual's first-order condition, and the computation of this policy is a complicated fixed-point problem. Here, the perfectly competitive policy may also serve as a close approximation. It has been noted that with increasing competition, the convergence to a competitive equilibrium is rapid. Grenadier (2002) provides an example where the value of an option to postpone investment reduces rapidly with competition with the option premium reducing to almost zero for 100 individuals (see his equation (32)). In the illustration of the previous section, we find that the exercise policy for $K >> 100$ is virtually identical to that of $K = 100$.

The analysis of this section also demonstrates the impact of the equilibrium exercise policies on the securities in the capital structure. As we noted, in the perfectly competitive equilibrium, for $V(T_1) \in [V^d, V^u]$, the stock price has zero volatility and remains fixed at X. Any change in firm value within this range results in an identical change in $\bar{W}$, irrespective of the specific securities that make up $\bar{W}$. This last observation underscores in a forceful way why the understanding of equilibrium exercise policies is important not only for the valuation of the warrant but also for the other securities in the capital structure.

Finally, the block exercise equilibrium for the single warrant may be considered a special case of the competitive equilibrium when $\bar{W} = 0$. Consider the set of values defined by $\mathbf{V}$ in (14). Substituting $\bar{W} = 0$, observe that there is only *one* firm value that is defined by $S(V, \delta) = X$, and this is $V = X$. This would suggest that partial exercise of warrants can occur in equilibrium only at $V(T_1) = X$. In fact, this precisely defines the block exercise policy of the single warrant as every $\delta \in [0, 1]$ is a equilibrium partial exercise strategy at this firm value. Lemma 1 provides the economic intuition behind why the block exercise may be considered a special case of the competitive equilibrium. When $\bar{W} \equiv 0$, every investor, irrespective of his holding, behaves as if the stock price is independent of his exercise. Therefore the equilibrium policy under the assumption that warrants are held by non-strategic naive investors replicates that of the competitive equilibrium.

3.4 Equilibrium Exercise Policies and Extraordinary Dividends

The equilibrium exercise policies derived in previous sections were determined by $\partial\bar{W}/\partial\delta^k$ being positive, i.e., warrant exercise resulted in wealth transfer from share- and warrant-holders to the other securities. Shareholders of the firm can limit or prevent such wealth transfer by giving an extraordinary dividend with part of the exercise proceeds. How is the equilibrium exercise policy affected by such a dividend policy?

Consider the illustration of Section 3.2, but assume that on exercise of warrants at T_1, the firm pays an extraordinary dividend per share of D to its shareholders that is equal to a fraction of the exercise proceeds, i.e., for a given y, $0 \le y \le 1$,

$$D = y\left(\frac{\sum_{k=1}^{K}\delta^k\alpha^k X}{1+\sum_{k=1}^{K}\delta^k\alpha^k}\right). \tag{16}$$

At T_1, $\bar{W}$ will be priced at the ex-dividend value of the firm, i.e., $V^+(T_1) = (V(T_1) + \sum_{k=1}^{K}\delta^k\alpha^k X - D)/(1+\sum_{k=1}^{K}\delta^k\alpha^k)$. The extraordinary dividend, if sufficiently high, significantly changes the equilibrium as it may no longer be true that exercising an at-the-money warrant will increase the value of $\bar{W}$.[7]

There are two interesting cases for the competitive equilibrium. First, in the range of firm values corresponding to $\mathbf{V}$, $\partial\bar{W}/\partial\delta$ is negative for any $\delta \in [0,1]$. From Lemma 2, there will not be partial exercise in equilibrium and the equilibrium policy will be in $\{0,1\}$. Second, it is also possible that $\partial\bar{W}/\partial\delta$ is positive only for $\delta \in [\delta^d, 1]$, where $0 < \delta^d \le 1$. In this case, we can still use Proposition 1 if we limit its application to this range of δ (the identical proof applies for $\delta \in [\delta^d, 1]$ instead of $\delta \in [0,1]$). There will be partial exercise in equilibrium over a range of firm values $[V^d, V^u]$, where $\delta(V^d) = \delta^d$ and $\delta(V^u) = 1$. However, in this case, we may expect that exercising none of the warrants is also an equilibrium strategy at V^d. Below, we provide illustrations of the competitive equilibrium policy for different levels of y. Figure 3 plots some of these equilibrium policies.

[7] This may be observed by direct computation. Differentiating $\bar{W}$ with respect to δ^k and simplifying using (1), we get,

$$\begin{aligned}\frac{\partial\bar{W}}{\partial\delta^k} &= \frac{\bar{\alpha}\alpha^k}{(1+\bar{\alpha}^+)(1+\sum_{k'=1}^{K}\delta^{k'}\alpha^{k'})^2}\left[\bar{W} - \Delta\left(V-(1-y)X\right)\right], \\ &= \frac{\bar{\alpha}^+\alpha^k}{(1+\bar{\alpha}^+)(1+\sum_{k'=1}^{K}\delta^{k'}\alpha^{k'})}\left[(1-\Delta)\bar{W} - \Delta(1+\sum_{k'=1}^{K}\delta^{k'}\alpha^{k'})(S-X) - y\Delta X\right],\end{aligned}$$

where, as previously defined, Δ is the "delta" of the call. Observe that for sufficiently high y, $\partial\bar{W}/\partial\delta^k < 0$ even when the warrant is exercised at-the-money, $S = X$. The proof is provided in Appendix B.

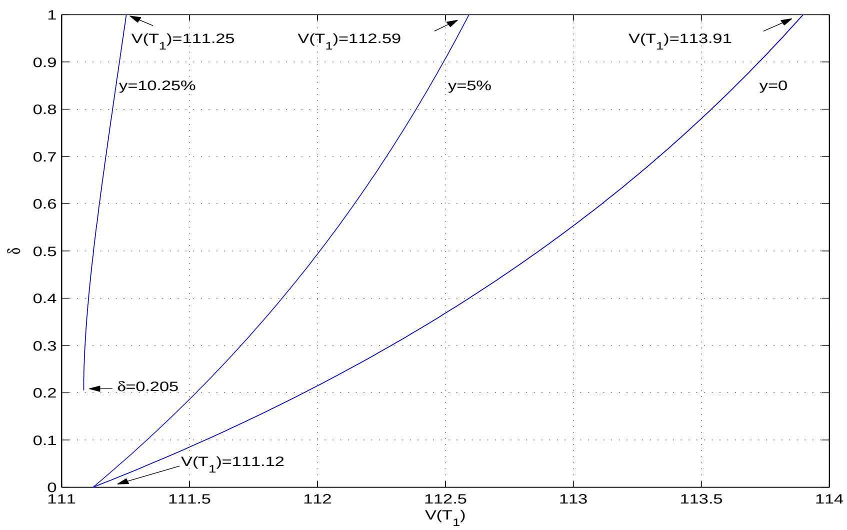

Figure 3: The graph plots the competitive equilibrium exercise policy δ when the firm pays an extraordinary dividend that is equal to $y\%$ of the cash proceeds from exercise. There are α units of an upstream warrant $\bar{W}$ of strike X and expiration date T_2. The parameters used to compute the exercise policy are: $\alpha = 1$, $\bar{\alpha} = 1$, $r = 0$, $X = 90$, $\bar{X} = 100$, $T_2 - T_1 = 5$.

- **Example 1:** $y = 5\%$.

  The competitive equilibrium exercise policy is,

$$\begin{aligned} \delta &= 1 \quad \text{if } V(T_1) > 112.59 \\ &= \hat{\delta}(V(T_1)) \quad \text{if } 111.12 \leq V(T_1) \leq 112.59 \\ &= 0 \quad \text{if } V(T_1) < 111.12 \end{aligned}$$

  where $0 \leq \hat{\delta}(V(T_1)) \leq 1$. As the dividend amount is small, $\partial\bar{W}/\partial\delta > 0$ for $V \in \mathbf{V}$ and $\delta \in [0, 1]$, and Proposition 1 applies. There is a unique competitive equilibrium, and every $\delta \in [0, 1]$ is attained in equilibrium. Compared to the equilibrium policy for $y = 0$, more warrants are partially exercised because the dividend reduces the transfer of wealth from warrant- and shareholders to the individuals who hold $\bar{W}$.

- **Example 2:** $y = 10.25\%$.

  The competitive equilibrium exercise policy is,

$$\begin{aligned} \delta &= 1 \quad \text{if } V(T_1) > 111.25 \\ &= \hat{\delta}(V(T_1)) \quad \text{if } 111.086 \leq V(T_1) \leq 111.25 \\ &= 0 \quad \text{if } V(T_1) \leq 111.086 \end{aligned}$$

where $0.205 \leq \hat{\delta}(V(T_1) \leq 1$. For firm values corresponding to $\mathbf{V}$, $\partial\bar{W}/\partial\delta$ is *not* strictly positive for $\delta < 0.205$ (in particular, at $V(T_1) = 111.086$, $\partial\bar{W}/\partial\delta < 0$ for $\delta < 0.205$). In this example, although the warrant may be partially exercised in equilibrium, there is no equilibrium policy where δ is in the interval $(0, 0.205)$. At $V(T_1) = 111.086$, there are two equilibria corresponding to $\delta = 0$ and $\delta = 0.205$, i.e., the equilibrium policy is no longer convex-valued.

- **Example 3:** $y = 30\%$.
  The competitive equilibrium exercise policy is,

$$\begin{aligned} \delta &= 0, \qquad V(T_1) \leq 106.52 \\ &= 1, \qquad V(T_1) \geq 106.52, \end{aligned}$$

  There is no partial exercise of warrants in equilibrium. Although this equilibrium is a block equilibrium, it differs from the block equilibrium for a single warrant. In the latter, the equilibrium policy is convex-valued, and every $\delta \in [0, 1]$ is an equilibrium policy at the exercise threshold. In this example, in contrast, at the firm value of 106.52, there are only two competitive equilibria corresponding to $\delta = 0$ and $\delta = 1$ and stock prices of \$87.02 and \$90, respectively. The corresponding values of $\bar{W}$ are \$19.50 and \$16.52, respectively. In the first equilibrium with the stock price of \$87.02, the warrant is out of the money and is not exercised. In the second equilibrium, the stock price is equal to \$90 only if all warrants are exercised.

In summary, seemingly straightforward corporate actions like an extraordinary dividend policy can result in equilibria with widely different security prices. The examples underline once again the importance of analyzing and determining equilibrium exercise policies.

3.5 The Competitive vs. the Monopolist Equilibrium

We can relate the competitive equilibrium strategy to the monopolist's equilibrium strategy. When comparing the equilibrium policies in Figure 2, we already noted that when there is partial exercise in equilibrium, fewer warrants are exercised by the monopolist. We make this statement precise. As the difference between the number of warrants exercised is related to the magnitude of wealth transfer to $\bar{W}$, we can relate the competitive exercise policy to the monopolist's exercise strategy through $\bar{W}$. Let the fraction of warrants partially exercised in equilibrium in the non-competitive equilbrium at $V(T_1) = V$ be denoted as δ^m. From equation

(6), $S(V,\delta^m) = X - \delta^m \partial S(V,\delta^m)/\partial\delta$. Substituting this value of S in (1), we get,

$$V = X - (1+\delta^m\alpha)\delta^m \frac{\partial S(V,\delta^m)}{\partial\delta} + \bar{W}(V,\delta^m). \tag{17}$$

At this same firm value, let δ^c be the number exercised in the competitive equilibrium. From Proposition 1, $S(V,\delta^c) = X_1$, and,

$$V = X_1 + \bar{W}(V,\delta^c). \tag{18}$$

Subtracting (17) from (18), we get,

$$\begin{aligned}\bar{W}(V,\delta^c) - \bar{W}(V,\delta^m) &= -(1+\delta^m\alpha_1)\delta^m \frac{\partial S(V,\delta^m)}{\partial\delta} \\ &= (1+\delta^m\alpha)\delta^m \frac{\partial \bar{W}(V,\delta^m)}{\partial\delta} \end{aligned} \tag{19}$$

where the last equality uses equation (5) to substitute for $\partial S/\partial\delta$. As $\bar{W}$ is strictly increasing in δ, it follows that $\delta^m < \delta^c$.[8] As we can compute δ^c and $\bar{W}(V,\delta^c)$ from Proposition 1, equation (19) also provides an alternative methodology for computing δ^m, and (upper) bound the warrant price.

4 Valuation and Implementation

We observed that the equilibrium exercise policy depends on the securities in the capital structure, and, in turn, the value of these securities will depend on the equilibrium exercise strategy. In deriving the equilibrium strategy, we need to consistently value all securities, including warrants, in the capital structure. We provide guidance on how this can be done given that we observe the stock price $S(t)$ but not $V(t)$.

For generality, we allow the warrants expiring at time T_1 to have $J \geq 1$ distinct strikes and dilution factors. Specifically, let there be α_j of each warrant W_j of strike X_j, $j = 1, ..., J$, $X_1 < X_2 < ... < X_J < \infty$. Correspondingly, we write equation (1) as,

$$V(t) = S(t) + \sum_{j=1}^{J} \alpha_j W_j(t) + \bar{W}(t), \tag{20}$$

[8] One way of observing this is to apply the Mean Value Theorem to the LHS of (19), which gives $\delta^m = \delta^c - (1+\delta^m\alpha)\delta^m \left(\frac{\partial\bar{W}(V,\delta^m)}{\partial\delta^m} / \frac{\partial\bar{W}(V,\tilde{\delta})}{\partial\tilde{\delta}}\right)$ for $\delta^m < \tilde{\delta} < \delta^c$.

Each of these warrants may be held by $K_j \geq 1$ individuals. The exercise strategy of the k'th individual holding a portion of W_j is denoted as δ_j^k.

4.1 Valuation when $\bar{W} \equiv 0$

If there are no other securities in the capital structure besides the stock and J warrants, the equilibrium exercise policy is a block equilibrium. This, in turn, determines the payoff of the warrant for each $V(T_1)$. The exercise threshold and payoff of the warrant are given as follows. For each of j warrants, $j = 1, .., J$:

1. *Exercise Threshold:* The warrant of strike X_j is fully exercised if $V(T_1) \geq \bar{V}_j$, where $\bar{V}_j = X_j + \sum_{j'=1}^{j} \alpha_{j'}(X_j - X_{j'})$, and expires worthless otherwise.

2. *Payoff:* The payoff of warrant of strike X_j is:

$$
\begin{aligned}
W_j(T_1) &= 0, \qquad V(T_1) < \bar{V}_j \\
&= \frac{1}{1+\bar{\alpha}_{j+l}} \left(V(T_1) - X_j^{j+l} \right), \qquad \bar{V}_{j+l} \leq V(T_1) < \bar{V}_{j+l+1},
\end{aligned} \tag{21}
$$

for $l = 0, 1, .., J - j$, $X_j^{j+l} \equiv X_j + \sum_{j'=1}^{j+l} \alpha_{j'}(X_j - X_{j'})$, and $\bar{\alpha}_j \equiv \sum_{j'=1}^{j} \alpha_{j'}$.

The exercise threshold can be derived by noting that Lemma 1 also applies when $J > 1$, i.e., a warrant is fully exercised if it is in-the-money. From (20), $S(T_1) \gtreqless X_j$ is equivalent to $V(T_1) \gtreqless \bar{V}_j$, where $\bar{V} = X_j + \sum_{j'=1}^{j} \alpha_{j'}(X_j - X_{j'})$. The payoff of the warrant in equation (21) follows by substituting the number of exercised warrants into equation (20) and re-arranging to express the payoff in terms of $V(T_1)$. For any $V(T_1)$, the payoff of the warrant depends on the total number of exercised warrants. As the number of in-the-money warrants depends on the level of $V(T_1)$, the slope of the payoff of the warrant changes as $V(T_1)$ increases, and more warrants are in-the-money. Over the range $0 < V(T_1) < \infty$, the payoff for the j'th warrant must be described separately for each of $J - j + 1$ segments. Despite the complicated payoff structure of the warrant, we can nevertheless derive a simple recursive valuation formula for the prices of the warrants that generalizes the single European warrant pricing formula of Galai and Schneller (1978).

Proposition 2 *Suppose the capital structure of the firm consists of the stock and $J \geq 1$ warrants of maturity T_1 and strikes X_j, $j = 1, 2, ..., J$. Then the value of these J warrants is* *given as follows. For $j < J$,*

$$W_j(t) = \frac{1}{1+\bar{\alpha}_j}\left(C(V(t), \bar{V}_j; T_1 - t) - C(V(t), \bar{V}_{j+1}; T_1 - t)\right) + W_{j+1}(t),$$

and, for $j = J$,

$$W_J(t) = \frac{1}{1+\bar{\alpha}_J} C(V(t), \bar{V}_J; T_1 - t),$$

where $\bar{V}_j$ is defined as

$$\bar{V}_j \equiv X_j^j = X_j + \sum_{j'=1}^{j} \alpha_{j'}(X_j - X_{j'}),$$

and $\bar{\alpha}_j \equiv \sum_{j'=1}^{j} \alpha_{j'}$, $C(V(t), X; T - t) = E_t^Q e^{-r(T-t)} \max(V(T_1) - X, 0)$.

Proof of Proposition 2: See Appendix.

The proof of the proposition uses the fact noted earlier that when all warrants have the same maturity, the payoff of a warrant is piecewise linear in $V(T_1)$ with the number of segments equal to the number of distinct warrants that are exercised. Because of this piecewise linearity, a warrant bull spread is equivalent to the corresponding (diluted) spread in call options. Recall that when there is a single warrant series, a warrant is equivalent to a diluted call on the value of the firm. The expression for a single warrant is a special case when one of the options in the option spread has a strike of infinity (and a price of 0), so that the existing single warrant pricing theory of equation (7) is subsumed in Proposition 2.[9]

4.2 Valuation with $\bar{W} \neq 0$

When the capital structure of the firm has other securities, $\bar{W} > 0$, we first need to compute the equilibrium exercise strategy given K_j, as demonstrated in Section 3. We assume below that the securities are valued under a competitive equilibrium and that the equilibrium exercise strategy is a partial exercise strategy,[10] although with modification the methodology may be

[9] If we substitute out the recursive structure of Proposition 2, we get,

$$W_j(t) = \frac{C(V(t), \bar{V}_j; T_1 - t)}{1+\bar{\alpha}_j} - \sum_{j'=j+1}^{J} \frac{\alpha_{j'} C(V(t), \bar{V}_{j'}; T_1 - t)}{(1+\bar{\alpha}_{j'})(1+\bar{\alpha}_{j'-1})}$$

A similar expression has also been independently noted by Darsinos and Satchell (2002).

[10] Proposition 1 can be generalized to allow for $J > 1$. Corresponding to a warrant of each strike X_j, there exists a competitive exercise policy defined by $S(T_1) = X_j$. This policy, $\hat{\delta}_j(V)$ can be computed by substituting $S(T_1) = X_j$ in equation (20), and solving the corresponding fixed point problem for each $\hat{\delta}_j \in [0, 1]$. The policy

extended to other equilibria.

Given $\delta_j(V(T_1))$, the payoff of each exercised warrant, $W_j = (S(T_1) - X)^+$, may be written in terms of $V(T_1)$ using equation (20). The payoff of W_j is,

$$W_j(T_1) = \frac{1}{1+\bar{\alpha}_J}\left(V(T_1) - \hat{X}_j(V(T_1))\right), \tag{22}$$

where

$$\hat{X}_j(V(T_1)) \equiv X_j + \sum_{j'=1}^{J} \delta_{j'}(V(T_1))\ \alpha_{j'}(X_j - X_{j'}) + \bar{W}(T_1), \tag{23}$$

and $\bar{\alpha}_J \equiv \sum_{j'=1}^{J} \delta_{j'}\alpha_{j'}$.

Equations (22) and (23) express the payoff of each warrant in terms of a *state-dependent adjusted strike*, $\hat{X}_j(V(T_1))$, that depends on the value of the firm at exercise. $\hat{X}_j$ accounts for the effect of both exercised warrants and downstream securities on the payoff. Unlike the case when $\bar{W} = 0$, the payoff is non-linear, and therefore, there does not exist any equivalence between warrant and option prices as in Proposition 2. Non-linearity arises from the dependence of the payoff on the value of upstream securities, which in turn depends on how many warrants are exercised at T_1.

Given the exercise strategy, the value of each warrant is computed numerically using equations (22) and (23) by taking expectations of the discounted payoff. Given the value of the warrants and other securities, we can compute the exercise strategy. Thus, when only $S(t)$ is observable, an iterative procedure will have to be implemented to ensure consistency of both exercise strategies and prices of the securities.

For illustration, consider once again our two-warrant example. Conditioned on an exercise policy of $\hat{\delta}(V(T_1))$, the payoff of W at T_1 is (from equation 22),

$$W(T_1) = \frac{1}{1+\hat{\delta}\alpha}\text{Max}(V(T_1) - \hat{X}, 0), \tag{24}$$

where $\hat{X} = X + \bar{W}(T_1)$. The value of $\bar{W}$ at T_1 is,

$$\bar{W}(T_1) = \frac{\bar{\alpha}}{1+\bar{\alpha}^+}C(V^+(T_1), \bar{X}; T_2 - T_1), \tag{25}$$

where $V^+ = (V(T_1) + \hat{\delta}X)/(1+\hat{\delta}\alpha)$, and $\bar{\alpha}^+ = \bar{\alpha}/(1+\hat{\delta}\alpha)$. Given these values of $W(T_1)$ and

exists if $\partial\bar{W}/\partial\delta_j > 0$ for $V \in \left\{V \mid V = X_j + \sum_{j'=1}^{j} \alpha_{j'}(X_{j'} - X_j) + \bar{W}(V, \delta_j(V)),\ \delta_j \in [0,1]\right\}$.

$\bar{W}(T_1)$, the prices of these warrants at $t < T_1$ is the solution to the set of equations,

$$\begin{aligned} V(t) &= S(t) + \alpha W(t) + \bar{W}(t), \\ W(t) &= \mathrm{E}_t^Q e^{-r(T_1 - t)} W(T_1), \\ \bar{W}(t) &= \mathrm{E}_t^Q e^{-r(T_1 - t)} \bar{W}(T_1). \end{aligned}$$

The expectation can be computed by usual numerical methods as, for example, using a binomial tree. We provide below an example of an iterative algorithm that can be used for valuation.

1. Start with an initial guess of $V(t)$ (say, $V(t) = S(t)$).

2. Determine the exercise policy, $\hat{\delta}(V(T_1))$, by iteratively solving equation (15), $V(\hat{\delta}) = X + \bar{W}(V(\hat{\delta}), \hat{\delta})$, for every $\hat{\delta} \in [0, 1]$. Store this vector, $\hat{\delta}(V(T_1))$.

3. Given $V(t)$, construct the grid of $V(T_1)$. Then, for each $V(T_1)$ and corresponding $\hat{\delta}(V(T_1))$, calculate the value of all securities at T_1, i.e., compute the value of $W(T_1)$ and $\bar{W}(T_1)$ using equations (24) and (25), respectively.

4. Calculate the values of $W(t)$ and $\bar{W}(t)$ by taking expectations of the discounted payoff under Q,

$$\begin{aligned} W(t) &= \mathrm{E}_t^Q e^{-r(T_1 - t)} W(T_1) \\ \bar{W}(t) &= \mathrm{E}_t^Q e^{-r(T_1 - t)} \bar{W}(T_1), \end{aligned}$$

5. Estimate the new value of $V(t) = S(t) + \alpha W(t) + \bar{W}(t)$, and iterate to convergence by repeating steps 2-4.

5 Implications for Existing Literature

Empirical tests in the literature have been based on the single warrant pricing theory underlying equation (7), and have ignored the potential impact of, and on, the other securities in the capital structure. Overall, existing empirical tests have had limited success in explaining warrant prices.

Lauterbach and Schultz (1990) investigate whether model warrant prices are consistent with observed market prices. Assuming in turn that the underlying firm value process is either lognormal or has a constant elasticity of variance, they estimate the average percentage absolute difference between model and market prices to be 13.5% and 11.3%, respectively (see their Table 7, pp. 1203). The pricing error is even higher in Schulz and Trautmann (1994), where after adjusting for an early exercise premium, the average percentage absolute error is 26% (Table 2, pp. 855). An implication of the theory that we develop here is that pricing errors should be higher, in the cross-section, for firms that have other securities in the capital structure. In Schulz and Trautmann's sample, two-thirds of the firms have more than one series of warrants outstanding. Separating the firms with more than one set of warrants from those with a single warrant series, we compute the average pricing error. The pricing error for firms with a single warrant series is 15.2%. In contrast, the pricing error for firms with more than one set of warrants is 27.8%. As expected, the standard warrant pricing theory is less accurate for pricing a warrant series when there are other securities, especially other warrants, in the capital structure.[11]

That the standard model does not price warrants accurately has implications for the ongoing debate on the cost of employee stock options (ESO) to shareholders (Hall and Murphy (2003) provide an overview of the debate). To the best of our knowledge, existing ESO models and empirical research have not estimated this cost after accounting for wealth transfers across different classes of securities.[12] Although the development of an ESO model that accounts for all of its special features is beyond the scope of this paper, we can illustrate the significance of wealth transfers by approximating the ESO by a European warrant. Assume that the firm already has stock options outstanding, and the firm gives a new set of options to its employees. The issuance of employee stock options does not bring in new cash, and we assume that the firm value does not change on issuance. Using the valuation model developed in the previous section, Table 1 computes the ratio of the change in stock price to the value of the new ESOs. The ratio indicates the cost borne by shareholders after accounting for wealth transfers from the older ESOs to the new ESOs.

For example, suppose the firm has existing in-the-money warrants of strike 80, corresponding to 15% of outstanding shares. The firm gives away additional near-money warrants corre-

[11]A number of studies have looked at stock price reaction of corporate actions that impact the warrant price. Schulz (1993) documents that when warrants are called by the firm, the stock return on the announcement date is - 3%. Howe and Wei (1993) investigate the announcement effect when the warrant maturity is extended, and find a cumulative abnormal return on days -1 and 0 to be 1.49%. It would be interesting to extend this analysis to understand the price reaction of other securities in the capital structure.

[12]Although most firms and some academic work simply use the Black-Scholes formula to estimate the cost of ESOs (eg. Yermack (1995)), there are also more sophisticated models that account for some of the special features of ESOs. For instance, Brenner, Sundaram and Yermack (2000) and Chance, Kumar and Todd (2000) consider the cost of the ESO by taking into account a reset option that accounts for possible strike and maturity changes. Ingersoll (forthcoming) provides a model to estimate the cost of the ESO after accounting for sub-optimal early exercise by risk-averse and non-diversified employees (see also Hall and Murphy (2000, 2002)). Bettis, Bizjack and Lemmon (2005) use data on exercise of ESOs to estimate the cost to shareholders using a model by Carpenter (1998), where the probability of termination/exercise of the ESO is given exogenously.

Table 1: Fraction of the cost of new warrants borne by existing shareholders. W is an existing warrant of maturity 5 years, dilution α and strike X. $\bar{W}$ is a new warrant of strike 95, maturity of 10 years, and dilution of $\bar{\alpha} = 2\%$. S is the stock price. The firm value per share is 100. The riskfree rate is $r = 5\%$ and the volatility is $\sigma = 40\%$. Issuance of $\bar{W}$ does not result in new cash proceeds to the firm.

X	α %	S	W	$\bar{W}$	Shareholders' Cost %
50	15	91.52	56.50	-	-
		90.59	55.58	53.86	87.2
	25	87.00	51.98	-	-
		86.20	51.20	49.81	80.4
80	15	93.43	43.79	-	-
		92.46	42.90	55.43	88.0
	25	89.93	40.28	-	-
		89.08	39.52	52.21	81.7
100	15	94.41	37.21	-	-
		93.41	36.41	56.29	88.5
	25	91.43	34.29	-	-
		90.55	33.54	53.53	82.4

sponding to 2% of outstanding shares. From Table 1, this warrant has a price of 55.43, and the total value of these warrants is 1.1% of the firm value per share of 100. On issuance, the stock price changes from 93.43 to 92.46. This reduction in the stock price of 0.97 is only 88% of the value of the new warrants. That is, stockholders bear only a fraction of the cost of the new warrants, with the remainder being paid for by the holders of W.

Table 1 suggests that the ongoing cost to shareholders of issuance of ESOs may be substantially different from those estimated by existing models. The impact of the wealth transfer is of comparable magnitude to the impact of other features of the ESO. For example, estimates of the increase in shareholder costs because of the resetting feature of ESOs range from 7-10% in Chance, Kumar, and Todd (2000) to a maximum of 18% in Brenner, Sundaram and Yermack (2000). In comparison, for reasonable parameter values, Table 1 suggests that the cost of the ESO to shareholders may be lower by 11.5-19.6% because of wealth-transfer from existing holders of ESOs.

6 Conclusion

The exercise of a warrant impacts other securities in the capital structure, besides the stock. If the net effect of exercise is that the longer maturity securities increase in value, then there is a transfer of wealth away from both existing shareholders and exercising warrant-holders (who are the new shareholders). We demonstrate that a warrant-holder adjusts the quantity exercised to mitigate this adverse impact on the value of his warrants. However, we also note, the extent to which the warrant-holder is successful in doing so is affected by the concentration of warrant-holding. In the extreme case, when warrants are competitively held by a large number of warrant-holders, the net payoff to exercise is zero for some range of firm values.

In summary, strategic and competitive considerations impact a warrant-holder's equilibrium exercise policy, and this in turn impacts the value of other securities in the capital structure. Without modeling the entire capital structure of the firm, it is not possible to fully understand the economic impact of dilution. For instance, in the existing European warrant theory when there is only one warrant and stock in the capital structure of the firm, the concentration of warrant-holding does not affect warrant exercise or value. On the contrary, we demonstrate that it plays an important role in determining the equilibrium exercise strategy.

Our observation of the importance of determining equilibrium policies in the context of the entire capital structure of the firm also applies in other contexts. For instance, the exercise policy and valuation for the American warrant under a perfectly competitive equilibrium [Constantinides (1984)] is only well understood when there is one warrant series in the capital structure of the firm. Even though the competitive equilibrium developed in this paper for the European warrant can be applied to American warrants of firms that do not pay dividends, it would be of interest to develop a general theory for the American warrant. This remains for future research.

Appendix A

Proof of Proposition 1:

Consider $V \in \mathbf{V}$, i.e., the set of firm values at T_1 defined by substituting $S(T_1) = X$ in equation (1),

$$V = X + \bar{W}(V, \delta), \tag{26}$$

where $\delta \in [0, 1]$.

Write (26) as

$$V(\hat{\delta}) = X + \bar{W}(V(\hat{\delta}), \hat{\delta}), \tag{27}$$

which is the inverse problem of determining V for a given $\delta = \hat{\delta}$, and is a fixed point problem. Uniqueness of the fixed point follows as $\partial\bar{W}/\partial V$ is strictly less than 1. Define firm values for which the corresponding value of $\hat{\delta}$ is 0 and 1 as $V^d \equiv V(0)$ and $V^u \equiv V(1)$, respectively. Continuity of V in $\hat{\delta}$ follows from the continuity of $\bar{W}(V, \hat{\delta})$. Equating the $\hat{\delta}$ derivative of both sides of (27), and noting that $\frac{\partial\bar{W}}{\partial\hat{\delta}}\,|_V > 0$ and that $\frac{\partial\bar{W}}{\partial V} < 1$, a rearrangement to compute $V'(\hat{\delta})$ shows it is always positive. By the Inverse Function Theorem, it follows that $\hat{\delta}(V)$ exists, and that it is increasing. Thus, $\hat{\delta}(V)$ defined by $S = X$ is an exercise policy establishing the main claim under part iii. Moreover, the uniqueness of $\hat{\delta}(V)$ allows us to identify $\hat{\delta}(V^d) = 0$ and $\hat{\delta}(V^u) = 1$. which proves A and B under iii. Parts C and D are addressed below.

Consider what happens when some exercise policy, δ, other than that specified by $\hat{\delta}(V)$, is used. In this case, equation (1) becomes

$$V = S + \delta\alpha\,(S - X) + \bar{W}(V, \delta). \tag{28}$$

Fixing V, subtract equation (27) from (28) to obtain

$$(1 + \delta\alpha)\,(S - X) = \bar{W}(V, \hat{\delta}) - \bar{W}(V, \delta). \tag{29}$$

Items C and D of part iii follow from (29) and the fact that $\bar{W}$ is an increasing function of δ.

To complete the proof of the claims under parts i and ii, subtract the defining equations for V^d and V^u from equation (28), for some V with the exercise policy δ. To prove part i (part ii is similar), We have

$$
\begin{aligned}
V - V^d - (1+\delta\alpha)(S-X) &= \bar{W}(V,\delta) - \bar{W}(V^d,0) &(30)\\
&= \bar{W}(V,\delta) - \bar{W}(V,0) + \bar{W}(V,0) - \bar{W}(V^d,0) &(31)\\
&= \frac{\partial \bar{W}(V,\delta)}{\partial \delta}\delta + \frac{\partial \bar{W}(\hat{V},0)}{\partial \hat{V}}\left(V - V^d\right), &(32)
\end{aligned}
$$

where the latter step involving $\hat{V}$ and δ uses the Mean Value Theorem. Since $\partial \bar{W}/\partial V < 1$ and $\bar{W}$ increases with δ, equation (32) implies $S < X$.

Proof of Proposition 2:

In the following proof, we adopt the notation $C_j \equiv C(V(t), \bar{V}_j; T_1 - t)$, and, without loss of generality, we assume the risk-free rate, r, to be equal to zero. Also, define $\bar{\alpha}_J \equiv \sum_{j=1}^{J} \alpha_j$.

First, note that $C_j = \int_{\bar{V}_j}^{\infty}(V(T_1) - \bar{V}_j)dQ$. Consider the J'th warrant, W_J. Given the payoff W_J, it immediately follows that $W_J = C_J/(1+\bar{\alpha}_J)$. Now consider the j'th warrant, $j < J$. From (21), the value of W_j under the risk-neutral distribution, Q, is,

$$
W_j(t) = \int_{X_j^j}^{X_{j+1}^{j+1}} \frac{V(T_1) - X_j^j}{1+\bar{\alpha}_j} dQ + \int_{X_{j+1}^{j+1}}^{X_{j+2}^{j+2}} \frac{V(T_1) - X_j^{j+1}}{1+\bar{\alpha}_{j+1}} dQ + \ldots + \int_{X_J^J}^{\infty} \frac{V(T_1) - X_j^J}{1+\bar{\alpha}_J} dQ. \quad (33)
$$

Therefore,

$$
W_j(t) - W_{j+1}(t) = \int_{X_j^j}^{X_{j+1}^{j+1}} \frac{V(T_1) - X_j^j}{1+\bar{\alpha}_j} dQ + \int_{X_{j+1}^{j+1}}^{X_{j+2}^{j+2}} \frac{X_{j+1}^{j+1} - X_j^{j+1}}{1+\bar{\alpha}_{j+1}} dQ + \ldots + \int_{X_J}^{\infty} \frac{X_{j+1}^J - X_j^J}{1+\bar{\alpha}_J} dQ. \quad (34)
$$

Consider the first term in equation (34),

$$
\begin{aligned}
\int_{X_j^j}^{X_{j+1}^{j+1}} \frac{V(T_1) - X_j^j}{1+\bar{\alpha}_j} dQ &= \frac{1}{1+\bar{\alpha}_j} C_j - \int_{X_{j+1}^{j+1}}^{\infty} \frac{V(T_1) - X_j^j}{1+\bar{\alpha}_j} dQ &(35)\\
&= \frac{1}{1+\bar{\alpha}_j} C_j - \frac{1}{1+\bar{\alpha}_j} C_{j+1} - \int_{X_{j+1}^{j+1}}^{\infty} (X_{j+1} - X_j) dQ, &(36)
\end{aligned}
$$

where the last equality uses the definition of C_{j+1}, and substitutes X_j^{j+1} for X_j^j, using the identity,

$$
X_{j+1}^{j+1} - X_j^j = (1+\bar{\alpha}_j)(X_{j+1} - X_j). \quad (37)
$$

Finally, the sum of the second to the last terms of equation (34) may be simplified by using the identity,

$$X_{j+1}^{k} - X_{j}^{k} = (1 + \bar{\alpha}_k)(X_{j+1} - X_j), \tag{38}$$

to give

$$\int_{X_{j+1}^{j+1}}^{X_{j+2}^{j+2}} \frac{X_{j+1}^{j+1} - X_{j}^{j+1}}{1 + \bar{\alpha}_{j+1}} dQ + ... + \int_{X_J}^{\infty} \frac{X_{j+1}^{J} - X_{j}^{J}}{1 + \bar{\alpha}_J} dQ = \int_{X_{j+1}^{j+1}}^{\infty} (X_{j+1} - X_j) dQ. \tag{39}$$

It follows from equations (36) and (39) that

$$W_j(t) = \frac{1}{1 + \bar{\alpha}_j}(C_j - C_{j+1}) + W_{j+1}(t). \tag{40}$$

Note that the statement of the Proposition uses the notation $\bar{V}_j \equiv X_j^j$.

Appendix B

This appendix provides derivations for the formulae for $\partial \bar{W}/\partial \delta^k$ used in footnote 7, equation (12), and footnote 6. To compute $\partial \bar{W}/\partial \delta^k$, recall, from (9) that $\bar{W} = \bar{\alpha}\frac{C(V^+)}{1+\bar{\alpha}^+}$ and $\bar{\alpha}^+ = \frac{\bar{\alpha}}{1+\sum_{k'}^{K}\delta^{k'}\alpha^{k'}}$, and from equation (4) that

$$V^+ = \frac{V + (1-y)\sum_{k'=1}^{K}\delta^{k'}\alpha^{k'}X}{1+\sum_{k'=1}^{K}\delta^{k'}\alpha^{k'}}.$$

Differentiating $\bar{\alpha}^+$ and V^+ with respect to δ^k, we get,

$$\frac{\partial \bar{\alpha}^+}{\partial \delta^k} = \frac{-\bar{\alpha}\alpha^k}{(1+\sum_{k'=1}^{K}\delta^{k'}\alpha^{k'})^2},$$

and

$$\begin{aligned}\frac{\partial V^+}{\partial \delta^k} &= \frac{(1+\sum_{k'=1}^{K}\delta^{k'}\alpha^{k'})(1-y)(\alpha^k X) - (\alpha^k)(V + (1-y)\sum_{k'=1}^{K}\delta^{k'}\alpha^{k'}X)}{(1+\sum_{k'=1}^{K}\delta^{k'}\alpha^{k'})^2} \\ &= \frac{-\alpha^k(V-(1-y)X)}{(1+\sum_{k'=1}^{K}\delta^{k'}\alpha^{k'})^2}\end{aligned}$$

We can now compute $\partial \bar{W}/\partial \delta^k$ by differentiating (9) (where we define $\Delta \equiv \partial C(V^+)/\partial V^+$),

$$\begin{aligned}\frac{\partial \bar{W}}{\partial \delta^k} &= \frac{-\bar{\alpha}}{(1+\bar{\alpha}^+)^2}\frac{\partial \bar{\alpha}^+}{\partial \delta^k}C(V^+) + \frac{\bar{\alpha}}{1+\bar{\alpha}^+}\Delta\frac{\partial V^+}{\partial \delta^k} \\ &= \frac{\bar{W}}{(1+\bar{\alpha}^+)}\frac{-\partial \bar{\alpha}^+}{\partial \delta^k} + \frac{\bar{\alpha}}{(1+\bar{\alpha}^+)}\Delta\frac{\partial V^+}{\partial \delta^k} \\ &= \frac{\bar{W}}{(1+\bar{\alpha}^+)}\frac{\bar{\alpha}\alpha^k}{(1+\sum_{k'=1}^{K}\delta^{k'}\alpha^{k'})^2} + \frac{\bar{\alpha}}{(1+\bar{\alpha}^+)}\Delta\frac{-\alpha^k(V-(1-y)X)}{(1+\sum_{k'=1}^{K}\delta^{k'}\alpha^{k'})^2}, \\ &= \frac{\bar{\alpha}^+\alpha^k}{(1+\bar{\alpha}^+)(1+\sum_{k'=1}^{K}\delta^{k'}\alpha^{k'})}\left[\bar{W} - \Delta(V-(1-y)X)\right]. \qquad (41)\end{aligned}$$

Now noting from equation (1) that $V-(1-y)X = (1+\sum_{k'=1}^{K}\delta^{k'}\alpha^{k'})(S-X) + yX + \bar{W}$, we can also write,

$$\frac{\partial \bar{W}}{\partial \delta^k} = \frac{\bar{\alpha}^+\alpha^k}{(1+\bar{\alpha}^+)(1+\sum_{k'=1}^{K}\delta^{k'}\alpha^{k'})}\left[(1-\Delta)\bar{W} - \Delta(1+\sum_{k'=1}^{K}\delta^{k'}\alpha^{k'})(S-X) - y\Delta X)\right]. \quad (42)$$

Equations (41) and (42) are used in footnote 7. Substituting $y=0$ in equations (41) and (42), respectively, we get $\partial \bar{W}/\partial \delta^k$ of equation (12) and $\partial \bar{W}/\partial \delta^k$ of footnote 6, respectively.

References

[1] Black, F. and M. Scholes, 1973, The pricing of options and corporate liabilities, Journal of Political Economy, 81, 637-59.

[2] Bergman, Y. Z., B. D. Grundy and Z. Wiener, 1996, General properties of option prices, Journal of Finance, 51 (5), 1573-1610.

[3] Bettis, J. C., J. M. Bizjak and M. L. Lemmon, 2005, Exercise behavior, valuation, and the incentive effects of employee stock options, Journal of Financial Economics 74, 445-470.

[4] Brennan, M. and E. S. Schwarz, 1977, Convertible bonds: Valuation and optimal strategies for call and conversion. Journal of Finance, 32, 1669-1715.

[5] Brenner, M., R. K. Sundaram and D. Yermack, 2000, Altering the terms of executive stock options, Journal of Financial Economics 57, 103-128.

[6] Carpenter, J., 1998, The exercise and valuation of executive stock options, Journal of Financial Economics 48, 127-158.

[7] Chance, D., M. R. Kumar, and R. B. Todd, 2000, The 'repricing' of executive stock options, Journal of Financial Economics 57, 129-154.

[8] Constantinides, G., 1984, Warrant exercise and bond conversion in competitive markets, Journal of Financial Economics, 13, 371-97.

[9] Constantinides, G. and R. Rosenthal, 1984, Strategic analysis of the competitive exercise of certain financial options, Journal of Economic Theory, 32, 128-138.

[10] Darsinos, T. and S. Satchell, 2002, On the valuation of warrants and executive stock options: Pricing formulae for firms with multiple warrants/executive options. Working paper, University of Cambridge.

[11] Dennis, J. P. and R. J. Rendleman, 2003, A model for valuing multiple executive stock options issued by the same company, Working paper, University of Virginia.

[12] Emanuel, D., 1983, Warrant valuation and exercise strategy, Journal of Financial Economics, 12, 211-235.

[13] Galai, D. and M. I. Schneller, 1978, Pricing of warrants and the value of the firm, Journal of Finance, 33, 1333-1342.

[14] Grenadier, Steven, 1996, The strategic exercise of options: Development cascades and overbuilding in real estate markets, Journal of Finance, 51, 1653-1679.

[15] Grenadier, Steven, 2002, Option exercise games: An application to the equilibrium strategies of firms, Review of Financial Studies, 15(3), 691-721.

[16] Hall, B. J. and K. J. Murphy, 2000, Optimal exercise prices for executive stock options, American Economic Review 90(2), 209-214.

[17] Hall, B. J. and K. J. Murphy, 2002, Stock options for undiversified executives, Journal of Accounting and Economics 33, 3-42.

[18] Hall, B. J. and K. J. Murphy, 2003, The trouble with stock options, Journal of Economic Perspectives 17(3), 49-70.

[19] Heston, S., 1993, A closed form solution for options with stochastic volatility with applications to bond and currency Options, Review of Financial Studies, 6(2), 327-343.

[20] Howe, J. and P. Wei, 1993, The valuation of warrant extensions, Journal of Finance 48(1), 305-314.

[21] Ingersoll, J., 1977, A contingent-claims valuation of convertible securities, Journal of Financial Economics, 4, 289-322.

[22] Ingersoll, J., 1987, *Theory of Financial Decision Making*, Rowman and Littlefield:NJ.

[23] Merton, R., 1973, Theory of rational option pricing, Bell Journal of Economics and Management Science, 4 (Spring 1973), 41-83.

[24] Ingersoll, J., The subjective and objective evaluation of incentive stock options, Journal of Business, forthcoming.

[25] Murphy, K., 1999, Executive compensation, in O. Ashenfelter and D. Card (eds.), Handbook of labor economics, Vol. 3, North-Holland, Amsterdam, 2485-2563.

[26] Lauterbach, B. and P. Schultz, 1990, Pricing warrants: An empirical study of the Black-Scholes model and its alternatives, Journal of Finance, 45, 1181-1209.

[27] Schultz, P, 1993, Calls of warrants: Timing and market reaction, Journal of Finance 48 (2) , pp. 681-696.

[28] Schulz, U. G. and S. Trautmann, 1994, Robustness of option-like warrant valuation, Journal of Banking and Finance (18), 841-859.

[29] Spatt, C.S. and F. P. Sterbenz, 1988, Warrant exercise, dividends and reinvestment policy, Journal of Finance, 43, 493-506.

[30] Williams, J. T., 1993. Equilibrium and options on real assets, Review of Financial Studies, 6, 825-850.

[31] Yermack, D., 1995, Do corporations award CEO stock options effectively? Journal of Financial Economics 39, 237-269.