# Dynamic Volatility Strategy with Recursive Utility

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**Abstract**

In this article, an analytical approximate solution method is given to provide investors with the means to make optimal consumption and portfolio choices with recursive utility in a complete market. The investment opportunity set is stochastic over time. The method is used to provide an exact determination of the unit elasticity of intertemporal substitution. An approximate solution method is derived in closed form for more general applications. The solution method is shown to provide the same solutions for cases with known analytical solutions, thus proving its effectiveness. Heston's (1993) stochastic volatility model is solved in detail as a practical and important illustration of the solution method. The market is complete with trading of derivatives, through either options or pure volatility derivatives such as variance swap. New insights gained from the solution method, such as the hedging demand for derivative securities due to the stochastic nature of price volatility, are detailed and discussed. Previous solutions have either excluded volatility trading or assumed the expected additive utility without intertemporal consumption. The model is calibrated to the S&P 500 index and VIX index. The impact of elasticity of intertemporal substitution is separated from that of risk aversion. Contrary to existing literature on the hedging demand of volatility, the effect of the elasticity of intertemporal substitution on hedging demand for derivatives is proven to be of first-order importance. The investment horizon effect on portfolio choice is also examined.

Keywords: Portfolio Choice, Recursive Utility, Martingale Approach, Stochastic Volatility, Analytical Solution

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# 1 Introduction

In a classical paper (Merton, 1971) Merton solved, in closed form, the optimal consumption and investment problem for investors. The solution included an expected additive utility (HARA utility) and a constant investment opportunity set. Recently, Merton's solution has been applied to a stochastic investment opportunity set with affine dynamics (e.g., Kim & Omberg, 1996; Liu, 1998; Liu & Pan, 2003; Wachter, 2002). Merton's problem has also been extended to more generalized preferences, the recursive utility, which nests the expected additive utility (e.g., Campbell & Viceira, 2002; Giovannini & Weil, 1989). Although Schroder and Skiadas (1999), who solved Merton's problem with recursive utility in a complete market setting with affine state variable processes, provide a more extensive approach, they showed that an analytical solution exists only for special parametrization of recursive utility. In this study, the existing literature is extended by exploring approximate analytical solutions for a general class of parametrization of recursive utility. Recursive utility, inspired by Koopmans (1960) and Kreps and Porteus (1978), was developed for general discrete-time multiperiod asset-pricing applications by Epstein-Zin (1989), the continuous limit of which was formulated by Duffie and Epstein (1992). In this article, the solution is based on continuous formulation of recursive utility.

Recursive utility can be considered as an extension of expected additive utility. The additive utility model is extremely restrictive, and found to be inconsistent with experimental evidence on choice under uncertainty (Plott, 1986). A drawback of additive utility is that the two psychologically separate concepts of risk aversion (the desire to stabilize consumption across states of nature) and the elasticity of intertemporal substitution (the desire to stabilize consumption over time) are constrained to be reciprocal of one another. Recursive utility allows for the separation of these two parameters; thus, it promotes a clean analysis of the comparative statics of risk. As pointed out by Dumas et al (1997), "even if it were true that, in the real world, each person's risk aversion were always exactly equal to the inverse of his/her elasticity of intertemporal substitution, it is still important to distinguish between the two concepts, in order to determine the size and direction of the effects of a change in the risks that investors face."

This article contributes to the existing literature by providing a solution method for general cases of recursive utility approximately. A stochastic volatility example is used to show the approximate solution that should satisfy investors with reasonable preferences. Specifically, the dynamic asset allocation problem is solved with derivative security as a trading instrument in Heston's (1993) stochastic volatility model. This example is of practical interest by itself and, in fact, was chosen based on interest in the demand behavior for various volatility products (such as VIX index and its related derivatives, OTC variance swaps, etc.), which have recently become popular in the marketplace. It has been well documented in the empirical literature that the variance risk premium of S&P 500 index is significant and negative (Bakshi & Kapadia, 2003; Bondarenko, 2004). This implies that investors who want long volatility will have to pay. One might argue that risk-averse investors are willing to accept a negative excess return because of the insurance provided by long volatility, such as put options. Bondarenko (2004) has shown that a wider class of investment strategies is exposed to volatility risk other than options trading. However, the exception is most hedge funds; most are short in volatility. Liu and Pan (2003) solved the problem of demand for volatility trading for CRRA investors without intertemporal consumption. They argue for more risk-averse investors as opposed to investors who prefer logarithmic utility; the demand for volatility is always negative. In this sense, hedge funds provide a service for investors who seek short volatility. In this article, Liu and Pan' solution to generalized preference is expanded in an effort to gain richer insights into the hedging demand for volatility products.

In a different setting, Chacko and Vicera (2005) solved a similar problem with recursive utility. They concentrated on the hedging demand of stocks due to stochastic volatility without the trading of volatility, i.e., the incomplete market case. Here their study is supplemented by a solution for recursive preference under a complete market scenario in a stochastic volatility setting.

Campbell and Vicera (2001) state that in a constant volatility setting, "the elasticity of intertemporal substitution is of second order importance for portfolio demand." In this article, it is shown that this statement is no longer valid in terms of volatility demand. In fact, in a stochastic volatility setting, investor's willingness to substitute consumption over time has first-order impact on demand for volatility trading. In addition, there is also a horizon effect on this demand behavior for volatility.

The remainder of the article is organized as follows. In the next section, the theoretical background is given as well as a general presentation and approach of the solution method. The solution method is then described in detail; the exact solution for unit elasticity of intertemporal substitution is presented first, followed by the derivation of the approximated solution for more general cases, which is based on the exact solution. An application in a stochastic volatility model is detailed and the comparative statics of volatility demand are shown for different investor preferences and different investment horizons. A conclusion and a discussion of future research directions complete the article.

# 2 The Intertemporal Consumption and Portfolio Choice Problem

## 2.1 Investment Opportunity Set and Pricing Kernel

We discuss in the setting of an Arrow-Debreu market. The general formulation of which can be found in, e.g., Duffie (1996). The *state price density process*, or pricing kernel $\phi_t$ is determined by

$$\frac{d\phi_t}{\phi_t} = -[rdt + \eta_t' dB_t] \tag{1}$$
$$\phi_0 = 1$$

where $\eta_t$ is the *market-price-of-risk* process.

We assume $n$ risky assets[2] with price process

$$\frac{dS_t}{S_t} = (r + \mu_t^R)dt + \sigma_t^R dB_t \tag{2}$$

where $\mu_t^R$ and $\sigma_t^R$ are progressively measurable processes valued in $R^n$ and $R^{n\times n}$. $r$ is the risk-free interest rate. $\mu_t^R$ is the expected excess return at time $t$. We define the *excess return process*

$$dR_t = \mu_t^R dt + \sigma_t^R dB_t$$

A trading strategy is any progressively measurable process, $\Psi_t$, valued in $R^n$. Given any initial wealth $W_0$, consumption plan $C_t$, and trading strategy $\Psi_t$, the corresponding wealth process $W_t$ is defined by the *budget constraint equation*:

$$dW_t = W_t(\Psi_t' dR_t + rdt) - C_t dt \tag{3}$$

with $W_{t=0} = W_0$.

## 2.2 Recursive Utility and Martingale Approach

In this paper, we follow the recursive utility formulation of Duffie-Epstein (1992). The intertemporal value function is defined recursively:

$$V_t = E_t[\int_t^T f(C_s, V_s)ds] \tag{4}$$

[2]In complete market with stochastic volatility, the risky assets include derivative securities that are exposed to volatility risk. In this paper, we assume pure volatility derivatives that are not exposed to asset price risk.

with terminal condition $V_T = 0$. $f(C_s, V_s)$ is a normalized aggregator of current consumption and continuation utility. The conventional additive intertemporal von Neumann-Morgenstern utility is obtained by restricting the aggregator to be

$$f(C_s, V_s) = u(C_s) - \beta V_s \tag{5}$$

We further restrict the recursive utility (**??**) to be *homothetic*, i.e., for any consumption processes $C'$ and $C$, and any $\lambda > 0$, we have

$$U(\lambda C') \geq U(\lambda C) \iff U(C') \geq U(C)$$

One class of homothetic recursive utility is the Kreps-Porteus utility (Kreps and Porteus 1978) generated by the aggregator $f$ of the form

$$f(C, V) = \frac{\beta}{1 - \frac{1}{\psi}}(1-\gamma)V[(\frac{C}{((1-\gamma)V)^{\frac{1}{1-\gamma}}})^{1-\frac{1}{\psi}} - 1] \tag{6}$$

where $\beta$ is the rate of time preference, $\gamma > 0$ is the relative risk aversion, and $\psi > 0$ is the elasticity of intertemporal substitution. It can be shown that if we set $\psi = \frac{1}{\gamma}$ in (**??**), we obtain expected additive utility of constant relative risk aversion (CRRA).

When $\psi = 1$, the normalized aggregator $f(C, V)$ becomes

$$f(C, V) = \beta(1-\gamma)V[\log(C) - \frac{1}{1-\gamma}\log((1-\gamma)V)] \tag{7}$$

Merton's problem is formulated by:

$$\max_{C,\Psi} E_0[\int_0^T f(C_s, V_s)ds]$$

with equation (**??**) as the intertemporal budget constraint.

In Merton (1971), the optimal investment problem is solved by dynamic programming. In this paper, we follow the stochastic duality approach, or the so-called *martingale approach*, developed by Pliska (1986), Cox and Huang (1989), and Karatzas et al (1987) for expected additive utility, and generalized by Duffie and Skiadas (1994) to recursive utility. The advantage of martingale approach is to transform the dynamic problem into a static one through Arrow-Debreu state price density process (or the pricing kernel) in complete market. Hence, this approach can be applied even to non-markovian setting.

Specifically, in our setting, we assume Markovian pricing kernel $\phi_t$ that satisfies equation (**??**). Any security price $P_t$ is related to its future payoff $D_s, (s \geq t)$ through

$$P_t = E_t[\int_t^T \frac{\phi_s}{\phi_t} D_s ds]$$

The dynamic budget constraint (**??**) can be interpreted as an asset whose price is $W_t$ and pays instantaneous dividend equal to optimal consumption $C_t$. Hence, the optimally invested wealth $W_t$ must satisfy

$$W_t = E_t[\int_t^T \frac{\phi_s}{\phi_t} C_s ds]$$

With this reinterpretation, the dynamic optimal problem can be transformed into the static problem

$$\max_C E_0[\int_0^T f(C_s, V_s) ds]$$

s.t.

$$W_0 = E_0[\int_0^T \phi_s C_s ds]$$

Note that in complete market, any consumption can be financed by certain self-financing trading strategy. Therefore, once we obtain the optimal consumption, we can uniquely solve the portfolio choice problem. The optimal consumption is solved through Lagrangian multiplier:

$$\max_C \{E_0[\int_0^T f(C_s, V_s) ds] - \lambda(E_0[\int_0^T \phi_s C_s ds] - W_0)\} \tag{8}$$

The first order condition of program (**??**) is obtained by the so-called *gradient utility approach*. For detailed exposition of gradient utility approach, refer to Duffie and Skiadas (1994). For our purpose, we sketch the approach as follows.

We first define a Hilbert functional space $\mathcal{C}$ consist of adaptive consumption process $\{C_t, 0 \leq t \leq T\}$ with inner product naturally defined as

$$< m_t, h_t > \equiv E[\int_0^T m_t h_t dt]$$

**Definition** *Utility gradient* is defined as

$$\bigtriangledown V_0(C_t; h) \equiv \lim_{\alpha \to 0} \frac{V_0(C_t; \alpha h) - V_0(C_t)}{\alpha} \tag{9}$$

It was shown in Duffie and Skiadas (1994) that there exists an element in $\mathcal{C}$, denoted as $m_t(C_t)$, such that

$$\bigtriangledown V_0(C_t; h) = E[\int_0^T m_t h_t dt] \tag{10}$$

Furthermore, for general recursive utility defined in (**??**),

$$m_t(C) = \exp(\int_0^t f_V(C_s, V_s(C)) ds) f_C(C_t, V_t(C)) \tag{11}$$

Therefore, the first order condition of the program (**??**) becomes

$$m_t(C) = \lambda \phi_t \tag{12}$$

In conventional expected additive case defined in (**??**), $m_t$ takes a particularly simple form, i.e.,

$$m_t(C) = e^{-\beta t} u'(C_t)$$

which is solved by Cox and Huang (1989). Notice that for expected additive utility, the utility gradient at given state-time $(C_t, t)$ is local, while in general recursive utility case, $m_t(C)$ depends on the whole path of consumption process up to time $t$. To proceed with our approximate solution, we summarize the relevant facts that we base our method on.

**Lemma 1** (Schroder and Skiadas 1999) *Suppose one of the following two conditions holds:*

*1.* $\psi = 1$, *or*

*2. if the risk-premium $\eta_t$ is constant.*

*Then*

$$\frac{C_t}{W_t} = [\int_t^T \exp(-\int_t^s q_\tau d\tau) ds]^{-1} \tag{13}$$

*where*

$$q = \psi\beta - (\psi - 1)(r + \frac{1}{2\gamma}\eta_t \cdot \eta_t)$$

*Proof: see Schroder and Skiadas (1999).*

Notice for both cases the consumption-wealth ratio is deterministic. Technically, once we solve for the optimal consumption process, the portfolio choice can be obtained directly from the consumption process. Intuitively, this indicates that the "consumption-saving" choice could be separated from the "portfolio selection" choice.[3] Our solution method is built upon this intuition.

# 3 Solution Method

In this section, we first show the solution method in special case for $\psi = 1$. We follow the method in Schroder and Skiadas (1999). We then solve the problem for general $\psi$. According to *Lemma 1*, for $\psi = 1$, consumption-wealth ratio is deterministic even for stochastic investment opportunity. Our solution method is based on the assumption that for general $\psi \neq 1$ the consumption-wealth ratio doesn't vary too much. In this sense, our approach is similar to loglinear approximation developed by Campbell (1993) and Campbell and Viceira (2002). In their approach, loglinear approximation is used for long-term investor only, while our approach extends to account for investment horizon effect. Moreover, we use a different formulation and a broader angle to examine the solution method.

[3]Notice that Merton (1971) indicated that this is true for CRRA utility with deterministic investment opportunity.

### 3.1 Unit Elasticity of Intertemporal Substitution: $\psi = 1$

When $\psi = 1$, the normalized aggregator $f(C, V)$ takes the form of equation (??), by differentiating utility gradient $m_t$ of (??), we have:

$$\frac{dm_t}{m_t} = f_V dt + \frac{df_C}{f_C} \tag{14}$$

Define auxiliary variable $X_t \equiv \ln f_C$, and restate the first order condition (??) as:

$$\frac{dm_t}{m_t} = \frac{\phi_t}{\phi_t}$$

we have

$$dX_t = -[(r + f_V)dt + \frac{1}{2}\eta_t \cdot \eta_t dt + \eta_t' dB_t] \tag{15}$$

Let $\alpha = 1 - \gamma$, and

$$f(C, V) = \beta(1 + \alpha V)[\log C - \frac{1}{\alpha}\log(1 + \alpha V)] \tag{16}$$

we have

$$\begin{aligned} f_V &= \beta\alpha[\log C - \frac{1}{\alpha}\log(1 + \alpha V)] - \beta \\ &= \beta\alpha[\log\frac{C}{1 + \alpha V} + (1 - \frac{1}{\alpha})\log(1 + \alpha V)] \end{aligned}$$

Notice that

$$f_C = \beta\frac{1 + \alpha V}{C} = e^X$$

We further define

$$1 + \alpha V_t = \exp(J_t + (1 - k_t)X_t) \tag{17}$$

and substitute the above equation into equation (??), we have

$$\begin{aligned} dX_t &= -[(\beta(1 - \alpha)k_t - \beta)X_t + \beta(\alpha - 1)J_t - B + \gamma + \frac{\eta_t \cdot \eta_t}{2}]dt - \eta_t' dB_t \\ &\equiv \mu_x dt - \eta_t' dB_t \end{aligned} \tag{18}$$

where we define $-B \equiv \alpha\beta \log \beta - \beta$ And

$$\frac{\alpha f(C,V)}{1+\alpha V} = f_V(C,V) + \beta$$

Let

$$C_t = \beta e^{-X_t}(1+\alpha V_t) = \beta e^{J_t - K_t X_t} \tag{19}$$

$$dJ_t = \mu_J dt + Z_t d\widetilde{B_t} = \mu_J dt + (1-k_t)Z_t \eta_t dt + Z_t dB_t$$

where we define

$$\widetilde{B_t} = B_t + \int_0^t (1-k_s)\eta_s ds$$

In order to find $\mu_J$ we use the fact that

$$\frac{\alpha}{1+\alpha V_t}[dV_t + f(C_t, V_t)dt)]$$

is a martingale, we have (for computation details, refer to Appendix)

$$\beta(1-\alpha)k_t^2 - \beta k_t - \dot{k}_t = 0,\ k_T = 1 \tag{20}$$

$$-\mu_J = (1-k_t)(B - r - k_t \frac{\eta_t \cdot \eta_t}{2}) + k_t\beta(\alpha-1)J_t - B + \beta + \frac{1}{2}Z_t \cdot Z_t \tag{21}$$

Hence we get the following Backward Stochastic Differential Equation for $J_t$,

$$\begin{aligned} dJ_t =& - \quad [(1-k_t)(B - r - k_t\frac{\eta_t \cdot \eta_t}{2}) + k_t\beta(\alpha-1)J_t - B + g_1 + \frac{1}{2}Z_t \cdot Z_t]dt + Z_t d\widetilde{B}_t \\ J_T =& 0 \end{aligned} \tag{22}$$

Notice that the above equation for $J_t$ doesn't involve $X_t$.

Once we solve for the process of $J_t$, we can solve for $X_t$ by substituting $J_t$ into equation (**??**). Then we can find the consumption process using

$$C_t = \beta e^{J_t - k_t X_t}$$

According to *Lemma 1*, for $\psi = 1$, we have

$$\frac{C_t}{W_t} = \frac{\beta}{1 - e^{-\beta(T-t)}}$$

The fact that $C/W$ ratio is deterministic implies that the stochastic term of $C_t$ and $W_t$ are the same, so the stochastic term of $W_t$ is equal to the portfolio holding. Once we solve for the optimal consumption, we can obtain the optimal portfolio. We defer the explicit expression for optimal portfolio to the next session, where the solution for $\psi = 1$ becomes a special case of *Proposition 1*.

## 3.2 Approximate Solution for $\psi \neq 1$

For $\psi \neq 1$, define

$$G \equiv \left(\frac{C}{((1-\gamma)V)^{\frac{1}{1-\gamma}}}\right)^{1-\frac{1}{\psi}}$$

the intertemporal aggregator (**??**) becomes

$$f(C,V) = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)V[G(C,V)-1] \tag{23}$$

First, let's consider the function form of value function $V$. Due to the homotheticity of the Kreps-Porteus utility generated by (**??**), the value function $V$ can be written as

$$V(Y,W,t) = I(Y,t)W^{1-\gamma}$$

This has been established in Duffie and Epstein (1992). Taking partial derivative with respect to $W$, we obtain

$$V_W = (1-\gamma)\frac{V}{W} \tag{24}$$

Furthermore, by using dynamic programming, Duffie and Epstein (1992) show that the consumption choice at each instant must satisfy the *envelop condition*:

$$f_C(C,V) = V_W \tag{25}$$

Condition (**??**) can be derived from first order condition of Hamilton-Jacobi-Bellman's equation. Take the first order derivative with respect to $C$ in equation (**??**), we have

$$f_C = \beta\frac{G}{C}(1-\gamma)V \tag{26}$$

By substituting (**??**) and (**??**) into the *envelop condition* (**??**), we have

$$\beta \frac{G(1-\gamma)V}{C} = (1-\gamma)\frac{V}{W} \Rightarrow \beta G = \frac{C}{W} = \exp(c-\omega) \tag{27}$$

where we define $c \equiv \ln C$, and $\omega \equiv \ln W$. In the spirit of loglinear approximation, we assume that the consumption-wealth ratio doesn't vary too much around its mean, combining with (**??**) we have

$$\beta G = \frac{C_t}{W_t} \equiv \exp(c_0 - \omega_0) \approx \exp(c_0 - \omega_0) + \exp(c_0 - \omega_0) \cdot [(c-\omega) - (c_0 - \omega_0)] \tag{28}$$

for some reference point of $c_0 - \omega_0$. Define

$$\begin{aligned} g_1 &\equiv \left(\frac{C_t}{W_t}\right)_0 = \exp(c_0 - \omega_0) \\ g_0 &\equiv g_1 - g_1 \ln g_1 \end{aligned}$$

we have

$$\beta G \approx g_0 + g_1 \log(\beta G) = (g_0 + g_1 \log \beta) + g_1 \log G \tag{29}$$

Then the intertemporal aggregator of (**??**) becomes:

$$\begin{aligned} f(C,V) &\approx \frac{1}{1-\frac{1}{\psi}}(1-\gamma)V[(g_0 + g_1 \log\beta) + g_1 \log G - \beta] \\ &= \frac{1}{1-\frac{1}{\psi}}(1-\gamma)V[g_1(1-\frac{1}{\psi})(\ln C - \frac{1}{1-\gamma}\log(1-\gamma)V) + g_0 + g_1 \ln\beta - \beta] \\ &= g_1(1-\gamma)V[\log C - \frac{1}{1-\gamma}\log((1-\gamma)V) + (\frac{g_0}{g_1} + \ln\beta - \frac{\beta}{g_1})/(1-\frac{1}{\psi})] \\ &= g_1(1-\gamma)V[\log C - \frac{1}{1-\gamma}\log((1-\gamma)V) + h_0] \end{aligned} \tag{30}$$

where we have defined $h_0 \equiv (\frac{g_0}{g_1} + \ln\beta - \frac{\beta}{g_1})/(1-\frac{1}{\psi})$.

There are a couple of ways to choose the *reference point* $g_1(t) \equiv \left(\frac{C_t}{W_t}\right)_0$. For example, in Campbell (1993) suggested using the unconditional mean of the consumption-wealth ratio. In Chacko and Viceira (2005), the reference point was chosen to be the deterministic limit of risk-premium. In this paper, we also use this approach, i.e., we approximate the $C/W$ ratio around the ratio when $\eta_t$ approaches its deterministic limit. According to *Lemma 1*,

$$g_1 \equiv \left(\frac{C_t}{W_t}\right)_0 = \frac{q}{1-e^{-q(T-t)}} \tag{31}$$

with

$$q = \psi\beta - (\psi-1)(r + \frac{1}{2\gamma}\left(\eta_t \cdot \eta_t\right)_d) \tag{32}$$

where $\left(\eta_t \cdot \eta_t\right)_d$ is the deterministic limit of the risk-premium squared.

Once we have approximated the intertemporal aggregator as (**??**), the procedure for solving the optimal consumption and portfolio selection problem becomes exactly the same as the case for $\psi = 1$, except for some redefinition of parameters. The details can be found in Appendix. In order to solve for an analytical form, we further restrict the dynamics of the investment opportunity process of $\eta_t$ to a class of affine process.

## 3.3 Investment Opportunity: Affine Dynamics

Analytical solution can be obtained by assuming affine structure of the pricing kernel. Affine model has been extensively used in interest rate term structure modelling (e.g., Duffie and Kan 1996). Later on, it has been used in portfolio selection problem to derive analytical form solution, e.g. Liu (1999), Wachter (2002), and Schroder and Skiadas (1999).

Let

$$\eta_t \cdot \eta_t = L_0(t) + L_1(t) \cdot Y_t \tag{33}$$

where

$$\begin{aligned} dY_t &= (K_0(t) + K_1^Y(t)Y_t)dt + \sigma_t dB_t \\ \sigma_t\sigma_t^T &= H_0(t) + \sum_{k=1}^{n} H_1^{(k)}(t)Y_t^k \end{aligned} \tag{34}$$

The main result is summarized in the following proposition.

**Proposition 1** *Let $k_t$ solves*

$$g_1\gamma k_t^2 - g_1 k_t - \dot{k}_t = 0,\ \ k_T = 1$$

*where $g_1$ is defined in (**??**). Then the portfolio holding is solved as:*

$$\Psi_t = k_t(\sigma_t^R\sigma_t^{R'})^{-1}\mu_t^R + (\sigma_t^{R'})^{-1}Z_t' \tag{35}$$

*with*

$$Z_t = \beta_t^T\sigma(Y_t, t)$$

*and $\mu^R$ and $\sigma^R$ defined in (**??**), and $\beta_t$ solves the following ODE:*

$$\dot{\beta}_t + K_1^{'T}(t)\beta_t + \frac{1}{2}\beta_t^T H_1(t)\beta_t - \rho_1(t) = 0,\ \ \beta(T) = 0 \tag{36}$$

*where*

$$\begin{aligned} K_1'(t) &= K_1(t) - M_1(t) + k_t g1(\alpha - 1) \\ K_0'(t) &= K_0(t) - M_0(t) \end{aligned}$$

*with $M_0(t)$ and $M_1(t)$ defined as*

$$(1 - k_t)\sigma_t\eta_t = M_0(t) + M_1(t)\cdot Y_t$$

*and $\rho_0(t)$ and $\rho_1(t)$ defined as*

$$\begin{aligned} \rho_0(t) &= -[(1-k_t)(B-r) + \alpha_t k_t g_1(\alpha-1) + (g_1 - B) - \frac{k_t(1-k_t)}{2}L_0(t)] \\ \rho_1(t) &= \frac{L_1(t)}{2}k_t(1-k_t) - k_t g_1(\alpha-1)\beta_t \end{aligned}$$

**Corollary 1** *For $\psi = 1$, the solution is given by Proposition 1 by setting $g_1 = \beta$, and the solution is exact.*

The first term in (**??**) is the myopic portfolio holding, the second term is the hedging demand due to changing investment opportunity. In the next session, we apply the solution method developed above to Heston (1993) stochastic volatility model which belongs to the affine class.

# 4 Application: Heston Stochastic Volatility Model

We consider an economy consisting of a risk-free asset with price process $P_t$, a risky asset with price process $S_t$. The risky asset follows a stochastic volatility diffusive process. In particular, we assume Heston (1993) model of stochastic volatility. To complete the market, we include a derivative security on the risky asset in our portfolio choice problem. Without loss of generality, we assume the additional derivative security is a pure volatility derivative without exposure to underlying price risk. For example, this could be a delta-hedged option position, "delta-neutral" straddle that's considered in Liu and Pan (2003), OTC variance swaps, or exchange traded volatility derivatives such as VIX futures, etc.

**Definition** Let $g(V)$ denote the price of the pure volatility derivative that is used for hedging, and $g_V \equiv \frac{\partial g}{\partial V}$ is the vega, and $\frac{g_V}{g}$ measures the per dollar exposure to aggregate market volatility.

The price processes follow:

$$\begin{aligned}
\frac{dP_t}{P_t} &= rdt \\
\frac{dS_t}{S_t} &= (r + \eta_1 V_t)dt + \sqrt{V_t}dB_t^1 \\
dV_t &= \kappa(\bar{V} - V_t)dt + \sigma_V \sqrt{V_t}(\rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2)
\end{aligned}$$

where $r$ is risk-free interest rate which can be time varying. $B^1$ and $B^2$ are independent Brownian motions. The variance of risky asset, $V_t$, follows a square-root process with mean reversion coefficient $\kappa$, long term mean $\bar{V}$, and volatility coefficient $\sigma_V$, which are all assumed to be constants.

In addition, the instantaneous correlation coefficient between $V_t$ and $S_t$ is $\rho$. $\rho$ is negative. The risk-premium of risky asset is assumed to be proportional to square root of variance. These two specifications correspond to two well-known effects of S&P500 index, i.e., the "leverage effect" and the "positive feedback effect", *resp.* These two effects are different explanations on the stylized fact that the price of market decreases tend to be accompanied by volatility increases.

The pricing kernel is specified as

$$\frac{d\phi_t}{\phi_t} = -[rdt + \eta_1\sqrt{V_t}dB_t^1 + \eta_2\sqrt{V_t}dB_t^2] \tag{37}$$
$$\phi_0 = 1$$

We are particularly interested in the property of hedging demand for the volatility risk $B_t^2$.

## 4.1 Exact Solution: $\psi = 1$

We summarize the exact solution for unit elasticity of intertemporal substitution, for the purpose to compare with our approximate solution.

**Proposition 2** *Under unit elasticity of intertemporal substitution, i.e., $\psi = 1$, the optimal volatility holding can be decomposed to myopic and hedging components:*

$$\begin{aligned}\Psi^V_{myopic} &= (k_t\frac{\eta_2}{\sigma_V\sqrt{1-\rho^2}})\frac{g}{g_V}\\ \Psi^V_{hedge} &= \beta_t\frac{g}{g_V}\end{aligned}$$

*where $\beta_t$ for $0 < t < T$ solves the following Ricatti equation:*

$$\dot{\beta}_t + K_1(t)\beta_t + \frac{1}{2}\sigma_V^2\beta_t^2 - \frac{\eta_1^2+\eta_2^2}{2}k_t(1-k_t) = 0,\ \ \beta(T) = 0 \tag{38}$$

*where*

$$k_t = \frac{1}{1-(1-\gamma)(1-e^{-\beta(T-t)})}$$

*and*

$$K_1(t) = -\kappa - (1-k_t)\sigma_V(\rho\eta_1 + \eta_2\sqrt{1-\rho^2}) - k_t\gamma\beta.$$

*Proof: see Appendix.*

Both hedging demand and myopic demand are proportional to the dollar value of unit exposure to volatility risk, which is measured by $\frac{g}{g_V}$. For long term investors,

$$\Psi^V_{hedge} \propto \beta_t = \frac{\delta}{\kappa_1 + \kappa_2} \qquad \text{as} \qquad T \to +\infty \tag{39}$$

where

$$\begin{aligned} \kappa_1 &= \kappa + (1 - \frac{1}{\gamma})\sigma_V(\rho\eta_1 + \eta_2\sqrt{1-\rho^2}) + \beta \\ \kappa_2 &= \sqrt{\kappa_1^2 - \delta\sigma_V^2} \\ \delta &= \frac{1-\gamma}{\gamma^2}(\eta_1^2 + \eta_2^2) \end{aligned}$$

In order to consider the impact of investment horizon $T$ on the portfolio hedging demand, we define the concept of *half life*.

**Definition** The *half life* $\tau$ for the solution of $\Psi^V_{hedge}$ to go to steady state, i.e.,

$$\beta_\tau \to \beta_{+\infty}$$

The *half life* is indicative of the investment horizon impact on the solution of $\beta_t$.

The property of Ricatti equation for $\beta_t$, equation (**??**), shows that

$$\tau = \frac{1}{\kappa_2} + \frac{1}{\beta}$$

Finally, we consider the relative importance of demand for volatility between hedge portfolio and myopic portfolio:

$$d_t \equiv \frac{\Psi^V_{hedge}}{\Psi^V_{myopic}} = \frac{\beta_t\sigma_V\sqrt{1-\rho^2}}{k_t\eta_2}$$

## 4.2 Solution for $\psi \neq 1$

Now we turn to the approximate solution for general value of $\psi \neq 1$. We pay special attention to hedging demand for the volatility risk. As discussed in Section 3, we approximate the solution around the deterministic case, where the investment opportunity set becomes constant. Therefore, equation (**??**) becomes

$$q = \psi\beta - (\psi - 1)(r + \frac{1}{2\gamma}\eta_1^2\bar{V}) \tag{40}$$

A direct computation based on *Proposition 1* can be found in Appendix. The result is summarized in the following proposition.

**Proposition 3** *For recursive utility with $\psi \neq 1$, the optimal volatility holding can be decomposed to myopic and hedging components:*

$$\Psi^V_{myopic} = (\frac{\eta_2}{\gamma\sigma_V\sqrt{1-\rho^2}})\frac{g}{g_V} \tag{41}$$

$$\Psi^V_{hedge} = \beta_t\frac{g}{g_V} \tag{42}$$

*where*

$$\dot{\beta}_t + K_1(t)\beta_t + \frac{1}{2}\sigma_V^2\beta_t^2 + \frac{1-\gamma}{2\gamma^2}(\eta_1^2 + \eta_2^2) = 0, \ \ \beta(T) = 0 \tag{43}$$

*with*

$$K_1(t) = -\kappa - (1 - \frac{1}{\gamma})\sigma_V(\rho\eta_1 + \eta_2\sqrt{1-\rho^2}) - g_1$$

*Proof: see Appendix.*

The hedging demand for long term investors is proportional to

$$\Psi^V_{hedge} \propto \beta_t = \frac{\delta}{\kappa_1 + \kappa_2} \qquad \text{as} \qquad T \to +\infty \tag{44}$$

where

$$\kappa_1 = \kappa + (1 - \frac{1}{\gamma})\sigma_V(\rho\eta_1 + \eta_2\sqrt{1-\rho^2}) + q \tag{45}$$

$$\kappa_2 = \sqrt{\kappa_1^2 - \delta\sigma_V^2} \tag{46}$$

$$\delta = \frac{1-\gamma}{\gamma^2}(\eta_1^2 + \eta_2^2) \tag{47}$$

The property of Ricatti equation for $\beta_t$, equation (**??**), shows that the *half life* $\tau$ for the solution of $\beta_t$ is

$$\tau = \frac{1}{\kappa_2} + \frac{1}{q} \tag{48}$$

where $q$ is defined in (**??**).

The relative importance of demand for volatility between hedge portfolio and myopic portfolio:

$$d_t \equiv \frac{\Psi^V_{hedge}}{\Psi^V_{myopic}} = \frac{\beta_t \sigma_V \sqrt{1-\rho^2}}{k_t \eta_2}$$

The diagram of $\beta_t$, half life $\tau$ and relative importance $d_t$ are shown in Figure **??**, **??** and **??**, *resp.*

## 4.3 Model Calibration and Comparative Statics

To examine the impact of both investor preferences and market opportunity on the optimal portfolio holding, we fix a set of base-case parameters for the model. In a companion paper Zhu (2005), we obtain these base-case parameters from a joint estimation of time series of S&P500 index and VIX index dating from Jan. 1990 to Dec. 2005. We set the long-run mean at $\bar{V} = (0.16)^2$, the rate of mean reversion at $\kappa = 5.14$, and the volatility coefficient $\sigma_V = 0.37$. The correlation between price and volatility risk is $\rho = -0.6658$. The stock risk premium $\eta_1 = 4$, and the volatility risk premium is $\eta_2 = -10$. These are in general agreement with the existing literature, e.g., Anderson et al (2002), Pan (2002).

For long-term investor, hedging demand for volatility is proportional to (**??**), i.e., $\frac{\delta}{\kappa_1+\kappa_2}$, with parameters defined in (**??**) to (**??**). The sensitivity of hedging demand for volatility, $\Psi^V_{hedge}$, to risk aversion $\gamma$ and elasticity of intertemporal substitution $\psi$ for long-term investors are shown in **Figure ??**.

The sign of hedging demand is always negative when $\gamma > 1$, and positive for $\gamma < 1$. The magnitude of hedging demand for volatility $\Psi^V_{hedge}$ decreases as $\gamma$ increases. In fact, risk-aversion $\gamma$ has much less impact on the hedging demand when $\gamma$ becomes larger, (e.g.,

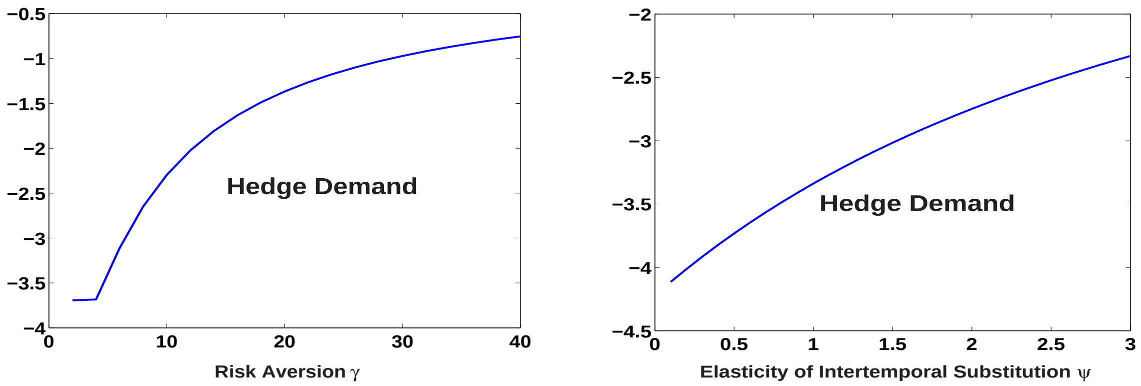


Figure 1: Hedging demand $\Psi^V_{hedge}$ v.s. $\gamma$ and $\psi$, with base case parameters defined as: risk aversion $\gamma = 3$, $\beta = 0.99$, $\eta_1 = 4$, $\eta_2 = -10$, $\sigma_V = 0.37$, $\rho = -0.6658$, $\kappa = 5.14$, $\bar{V} = 0.0267 = 16\%^2$.

$\gamma > 20$). $\psi$ doesn't affect the myopic demand for volatility, but it has first order impact on $\Psi^V_{hedge}$. In fact, $\Psi^V_{hedge}$ increases as $\psi$ decreases. Intuitively, lower elasticity of intertemporal substitution, i.e., less willing to substitute for future consumption, there would be higher hedging demand for volatility exposure.

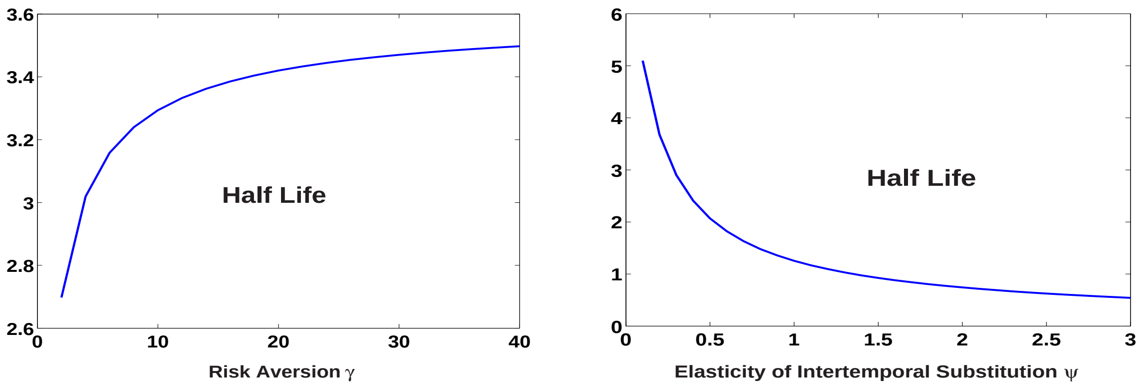


Figure 2: Half life $\tau$ v.s. $\gamma$ and $\psi$, with base case parameters defined as: risk aversion $\gamma = 3$, $\beta = 0.99$, $\eta_1 = 4$, $\eta_2 = -10$, $\sigma_V = 0.37$, $\rho = -0.6658$, $\kappa = 5.14$, $\bar{V} = 0.0267 = 16\%^2$.

Second, we consider the investment horizon effect on hedging demand for volatility. Notice that Chacko and Viceria (2005) considers long-term investment horizon where $T \to +\infty$. In addition to the steady state approximation results, we provide the horizon effect on the optimal portfolio problem when investment horizon is near. This can be obtained either through numerical approximation or analysis on the half life for $\beta_t$ characterized by equation (??). **Figure ??** shows the half life for $\Psi^V_{hedge}$, the hedging demand for volatility, to go to steady state. The magnitude of the half life is indicative of different behavior for long or short term investors. The elasticity of intertemporal substitution has much bigger impact on the horizon effect than risk aversion does. Specifically, for investors who are less willing to substitute for future consumptions, the optimal holding of volatility for hedging purpose has higher half life, hence horizon effect has bigger impact. In addition, horizon effect has bigger impact on more risk averse investors.

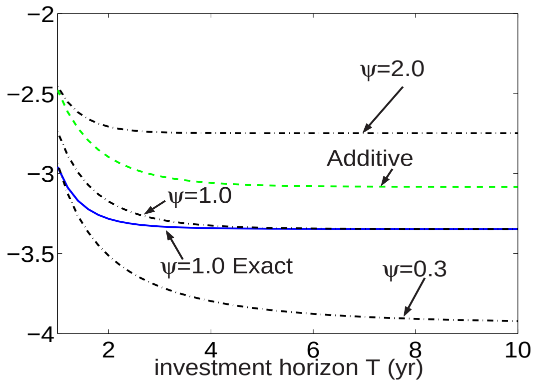


Figure 3: Hedging demand $\Psi^V_{hedge}$, with base case parameters defined as: risk aversion $\gamma = 3$, $\beta = 0.99$, $\eta_1 = 4$, $\eta_2 = -10$, $\sigma_V = 0.37$, $\rho = -0.6658$, $\kappa = 5.14$, $\bar{V} = 0.0267 = 16\%^2$.

Third, we shed some light on how the approximation method performs over time. Specifically, we compare the approximate solution to the exact solution when available, i.e., when $\psi = 1$. The comparison is shown in **Figure ??**. In this case, the approximation works very well. The intuitive reason why the approximation works well around $\psi = 1$ is due to the fact that there are two cases in which the consumption-wealth ratio is deterministic. One is the case with constant opportunity set, and another is the case with $\psi = 1$. Moreover, the consumption-wealth ratio coincides under these two cases. Our solution is an approximation assuming the volatility coefficient $\sigma_V$ to be small, which corresponds to the constant opportunity case. It's natural for the approximate solution to be exact when $\psi = 1$.

We also include the solution for expected additive CRRA utility in both **Figure ?? and ??**. The analytical solution for additive CRRA utility is provided in Appendix. We can see that holding risk aversion $\gamma$ constant, investors with lower $\psi$ will demand more (negative) volatility exposure for hedging purpose than additive investors. On the other hand, investors with higher $\psi$ may demand less (negative) volatility exposure.

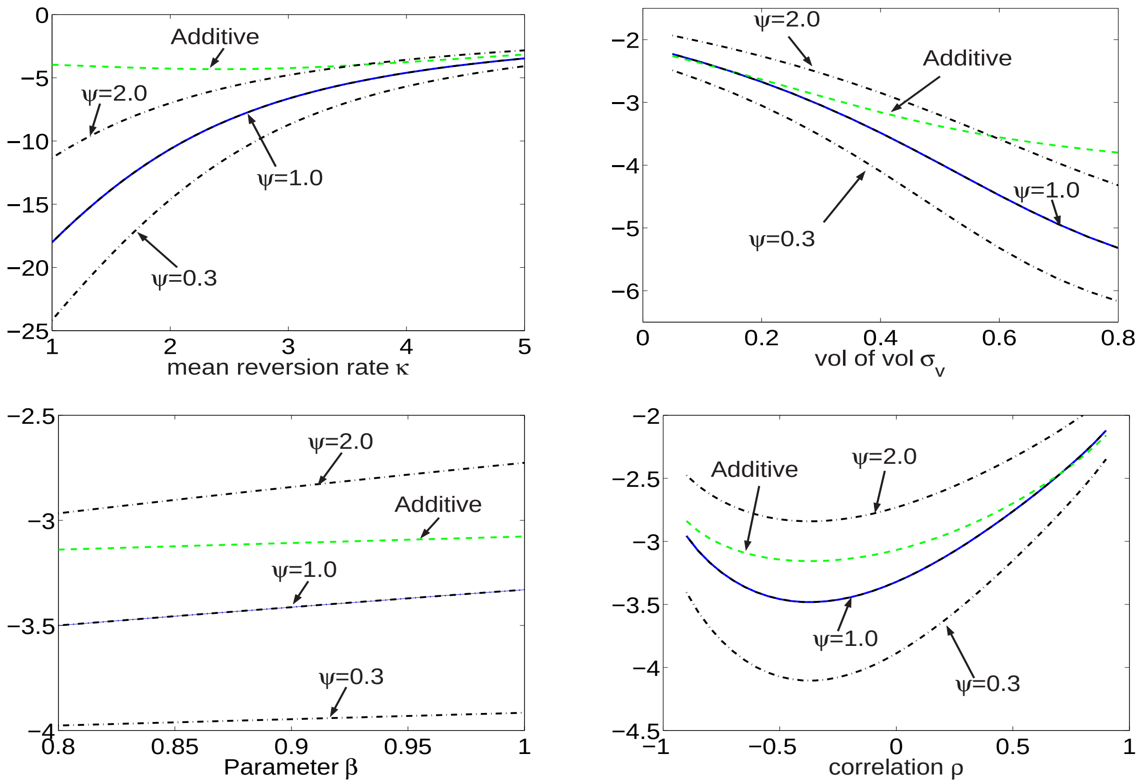


Figure 4: Hedging demand $\Psi^V_{hedge}$ for long term investors, with base case risk aversion $\gamma = 3$, $\eta_1 = 4$, $\eta_2 = -10$, $\sigma_V = 0.37$, $\rho = -0.6658$, $\kappa = 5.14$, $\bar{V} = 0.0267 = 16\%^2$.

Moreover, comparative statics for $\Psi^V_{hedge}$ with varying parameters are presented in **Figure ??**. Both lower mean reversion rate $\kappa$ and higher volatility coefficient $\sigma_V$ lead to higher demand for negative volatility exposure. The hedging demand for volatility is higher for lower absolute value of correlation between stock return and its volatility. In extreme case, if stock return and volatility is perfectly correlated, either positive or negative, volatility could be considered as redundant to stock, and wouldn't serve the purpose for hedging. It's interesting to observe that neither the correlation $\rho$ nor the volatility risk premium $\eta_2$ changes the sign of volatility holding. In addition, the discount factor $\beta$ doesn't have much impact on hedging demand for volatility.

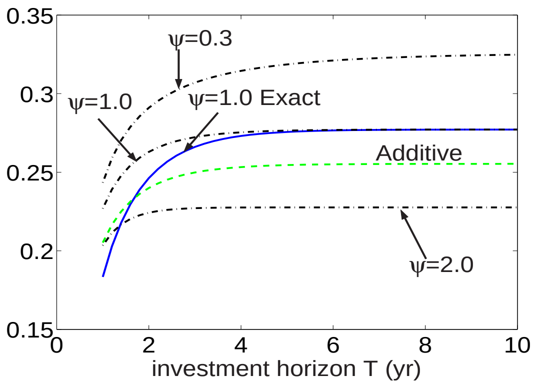


Figure 5: Relative importance Hedging demand $\Psi^V_{hedge}$ to myopic demand $\Psi^V_{myopic}$, with base case parameters defined as: risk aversion $\gamma = 3$, $\beta = 0.99$, $\eta_1 = 4$, $\eta_2 = -10$, $\sigma_V = 0.37$, $\rho = -0.6658$, $\kappa = 5.14$, $\bar{V} = 0.0267 = 16\%^2$.

Finally, we consider the myopic demand for volatility, $\Psi^V_{myopic}$. $\Psi^V_{myopic}$ is the same demand function for a risky asset (the volatility exposure in our case) by a single-period mean-variance optimizer. It's natural that $\Psi^V_{myopic}$ doesn't depend on $\psi$, which measures the preference for consumption substitution over time. Moreover, $\Psi^V_{myopic}$ is inversely proportional to $\gamma$, and it doesn't depend on time explicitly. **Figure ??** shows the relative importance of the hedging demand to myopic demand over time. As shown in equation (**??**), the sign of $\Psi^V_{myopic}$ is determined by volatility risk premium $\eta_2$. Negative volatility risk premium implies negative myopic demand.

# 5 Concluding Remarks

In this article, a solution method for the optimal consumption and portfolio selection problem for recursive utility with stochastic investment opportunity was explored. Previous studies have provided analytical solutions for investors with unit elasticity of intertemporal substitution of consumption and expected additive CRRA utility under the assumption of an affine pricing kernel. The existing literature has now been extended by this proposed method, which is a direct extension of the log-linear approximation method first developed by Campbell (1993). For a long investment horizon, the approach detailed here leads to the same analytical results of log-linear approximation. As in log-linear approximation, the approximation solution method is based on the assumption that the optimal consumption-wealth ratio does not vary too much around its unconditional mean.

To illustrate new insights that may be gained from the proposed method, it was used as an application in Heston's (1993) stochastic volatility model in complete market, with special attention paid to the characterization of the hedging demand for volatility exposure. Optimal portfolio demand on volatility is a combination of two components, a myopic (or mean-variance component) and an intertemporal hedging component. The relative importance of the two-demand component for volatility exposure is demonstrated through calibration of the joint dynamics of stock market returns and the volatility process using the joint data of the S&P 500 and the VIX volatility index from 1990-2005.

From the comparative statics of volatility trading both for myopic demand and for hedging demand, it is clear that the elasticity of intertemporal substitution has first-order effect on the demand for volatility exposure contrary to popular beliefs. Careful research into the demand function for volatility may shed light on the large volatility risk premium, the volatility-related market innovation, such as variance swaps and VIX futures as well as hedge funds' exposure to volatility.

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# A Appendix: Proof of Proposition 1

We follow the derivation from Duffie and Skiadas, and Schroder and Skiadas (1999), we have

$$m_t(t): \qquad \bigtriangledown V_0(C;h) =< m(C), h >$$

$$-\frac{d\phi_t}{\phi_t} = r_t dt + \eta_t' dB_t$$

First order condition:

$m_t(c) = \lambda\phi_t$ on $\{C_t > 0\}$ for a.e. $t \in [0,T]$

$m_t(c) \leq \lambda\phi_t$ on $\{C_t = 0\}$ for a.e. $t \in [0,T]$

$V_t(C) = E_t[\int_t^T f(s, C_s, V_s(C))ds]$, $t \in [0,T)$

$m_t(C) = \exp(\int_0^t f_V(s, C_s, V_s(C))ds) f_C(t, C_t, V_t(C))$

$$\begin{cases} -\frac{d\phi_t}{\phi_t} = r_t dt + \eta_t' dB_t \\ f_C(t, I(t,x,v), v) = e^x \\ dX_t = -[f_V(t, C_t, V_t(C) + r_t + \frac{1}{2}\eta_t'\eta_t]dt - \eta_t' dBt \\ V_t = E_t[\int_t^T f(s, I(s, X_s, V_s), V_s), V_s)ds] \\ W_t = E_t[\int_0^t \phi_s \cdot I(s, X_s, V_s)ds] \end{cases}$$

SDU:$f(C,V) = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)V[(\frac{C}{((1-\gamma)V)^{\frac{1}{1-\gamma}}})^{1-\frac{1}{\psi}} - 1]$

$$\psi \to 1: f(C,V) = \beta(1-\gamma)V[\log(C) - \frac{1}{1-\gamma}\log((1-\gamma)V)]$$

For $\psi \neq 1$, we have

$$\begin{aligned} f(C,V) &= \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)V[(\frac{C}{((1-\gamma)V)^{\frac{1}{1-\gamma}}})^{1-\frac{1}{\psi}} - 1] \\ &\equiv \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)V[G(C,V) - 1] \end{aligned} \tag{49}$$

According to Bellman's equation, we have

$$\left.\begin{array}{l} f_C = V_W \Rightarrow f_C = \beta\frac{G}{C}(1-\gamma)V \\ \\ V = I \cdot \frac{W^{1-\gamma}}{1-\gamma}, V_W = (1-\gamma)\frac{V}{W} \end{array}\right\} \Rightarrow \beta\frac{G(1-\gamma)V}{C} = (1-\gamma)\frac{V}{W} \tag{50}$$

Hence,

$$\beta G = \frac{C}{W} = \exp(c-\omega) \tag{51}$$

Loglinear approximation:

$$\beta G \approx g_0 + g_1\log(\beta G) = (g_0 + g_1\log\beta) + g_1\log G \tag{52}$$

Now the Stochastic Differential Utility of (**??**) becomes:

$$\begin{aligned} f(C,V) &\approx \frac{1}{1-\frac{1}{\psi}}(1-\gamma)V[(g_0+g_1\log\beta)+g_1\log G-\beta] \\ &= \frac{1}{1-\frac{1}{\psi}}(1-\gamma)V[g_1(1-\frac{1}{\psi})(\ln C-\frac{1}{1-\gamma}\log(1-\gamma)V)+g_0+g_1\ln\beta-\beta] \\ &= g_1(1-\gamma)V[\log C-\frac{1}{1-\gamma}\log((1-\gamma)V)+(\frac{g_0}{g_1}+\ln\beta-\frac{\beta}{g_1})/(1-\frac{1}{\psi})] \\ &= g_1(1-\gamma)V[\log C-\frac{1}{1-\gamma}\log((1-\gamma)V)+h_0] \end{aligned} \tag{53}$$

where we have defined $h_0 \equiv (\frac{g_0}{g_1}+\ln\beta-\frac{\beta}{g_1})/(1-\frac{1}{\psi})$.

Let $\alpha = 1-\gamma$,

$$f(C,V) = g_1(1+\alpha V)[\log C - \frac{1}{\alpha}\log(1+\alpha V)+h_0] \tag{54}$$

$$\begin{aligned} f_C &= g_1\frac{1+\alpha V}{C} = e^X \\ C &= g_1(1+\alpha V)e^{-X} \end{aligned}$$

We have

$$f_V = g_1(1-\gamma)[\log C - \frac{1}{1-\gamma}\log(\epsilon+(1-\gamma)V)+h_0] - g_1 \tag{55}$$

And

$$\frac{\alpha f(C,V)}{1+\alpha V} = f_V(C,V) + g_1$$

Let

$$C_t = g_1 e^{-X_t}(1+\alpha V_t) = g_1 e^{J_t - K_t X_t} \tag{56}$$

Then we have

$$1 + \alpha V_t = \exp(J_t + (1-k_t)X_t) \tag{57}$$

$$\begin{aligned} f_V &= g_1\alpha[\ln g_1 + J_t - k_t X_t - \frac{1}{\alpha}(J_t + (1-k_t)X_t) + h_0] - g_1 & (58)\\ &= (g_1(1-\alpha)k_t - g_1)X_t + g_1(\alpha-1)J_t - B & (59)\end{aligned}$$

where we define $-B \equiv \alpha g_1(\ln g_1 + h_0) - g_1$. Therefore, we have

$$\begin{aligned} dX_t &= -[(g_1(1-\alpha)k_t - g_1)X_t + g_1(\alpha-1)J_t - B + \gamma + \frac{\eta_t\cdot\eta_t}{2}]dt - \eta_t' dB_t \\ &\equiv \mu_x dt - \eta_t' dB_t \\ dJ_t &= \mu_J dt + Z_t d\widetilde{B_t} = \mu_J dt + (1-k_t)Z_t\eta_t dt + Z_t dB_t \end{aligned}$$

$\widetilde{B_t} = B_t + \int_0^t (1-k_s)\eta_s dS$

$$\begin{aligned} \frac{\alpha dV_t}{1+\alpha V_t} &= [\mu_J + (1-k_t)Z_t\eta_t + (1-k_t)\mu_x - k_t X_t + \frac{1}{2}(Z_t - (1-k_t)\eta_t')^2]dt \\ &+ (Z_t - (1-k_t)\eta_t')dB_t & (60)\\ \frac{\alpha f(C,V)}{1+\alpha V}dt &= [g_1(1-\alpha)k_t - g_1]X_t dt + [g_1(\alpha-1)J_t + \alpha g_1(\ln g_1 + h0)]dt & (61)\end{aligned}$$

$$\frac{\alpha}{1+\alpha V_t}[dV_t + f(C_t,V_t)dt)] = [\mathcal{A}_t X_t + \mathcal{B}_t]dt + [Z_t - (1-k_t)\eta_t']dB_t \tag{62}$$

$$\begin{aligned} \mathcal{A}_t =& g_1(1-\alpha)k_t^2 - g_1 k_t - \dot{k}_t \\ \mathcal{B}_t =& \mu_J + (1-k_t)(B - r - k_t\frac{\eta_t\cdot\eta_t}{2}) + k_t g_1(\alpha-1)J_t - B + g_1 + \frac{1}{2}Z_t\cdot Z_t \end{aligned} \tag{63}$$

$$g_1(1-\alpha)k_t^2 - g_1 k_t - \dot{k}_t = 0, \;\; k_T = 1 \tag{64}$$

$$-\mu_J = (1-k_t)(B - r - k_t\frac{\eta_t \cdot \eta_t}{2}) + k_t g_1(\alpha-1)J_t - B + g_1 + \frac{1}{2}Z_t \cdot Z_t \tag{65}$$

$$\begin{aligned} dJ_t = -[(1-k_t)(B - r - k_t\frac{\eta_t \cdot \eta_t}{2}) + k_t g_1(\alpha-1)J_t - B + g_1 + \frac{1}{2}Z_t \cdot Z_t]dt + Z_t d\widetilde{B}_t \\ J_T = 0 \end{aligned} \tag{66}$$

Under this assumption, we can solve (**??**) using Feymann-Kac formula by postulating a solution of affine form

$$J_t = \alpha_t + \beta_t \cdot Y_t \tag{67}$$

According to Schroder and Skiadas (1999) Lemma A1, we have the following Feyman-kac formula for Backward stochastic equation (**??**):

$$\begin{aligned} \exp(J_t) &= \widetilde{E}_t[\exp(\int_t^T ds(1-k_t)(B - r - k_t\frac{\eta_t \cdot \eta_t}{2}) + k_t g_1(\alpha-1)J_t - B + g_1)] \\ &\equiv \widetilde{E}_t[\exp(-\int_t^T R(Y_s, s)ds)] \end{aligned} \tag{68}$$

Where

$$\begin{aligned} dY_t &= (K_0(t) + K_1^Y(t)Y_t)dt + \sigma_t d\widetilde{B}_t \\ \sigma_t\sigma_t^T &= H_0(t) + \sum_{k=1}^n H_1^{(k)}(t)Y_t^k \end{aligned} \tag{69}$$

$Y_t$ can be expressed in terms of $B_t$:

$$dY_t = (K_0(t) + K_1^Y(t)Y_t + (1-k_t)\sigma_t\eta_t)dt + \sigma_t dB_t$$

Let

$$J_t = \alpha_t + \beta_t \cdot Y_t \tag{70}$$

$$\eta_t \cdot \eta_t = L_0(t) + L_1(t) \cdot Y_t \tag{71}$$

Using the formula in Duffie et al (2000), we obtain the following ODE for $\alpha_t$ and $\beta_t$

$$\dot{\beta}_t + K_1^{YT}(t)\beta_t + \frac{1}{2}\beta_t^T H_1(t)\beta_t - \rho_1(t) = 0,\ \beta(T) = 0 \tag{72}$$

$$\dot{\alpha}_t + K_0(t)\beta_t + \frac{1}{2}\beta_t^T H_0(t)\beta_t - \rho_0(t) = 0,\ \alpha(T) = 0 \tag{73}$$

where we define

$$R(Y_t, t) = \rho_0(t) + \rho_1(t) \cdot Y_t$$

$$\rho_0(t) = -[(1-k_t)(B-r) + \alpha_t k_t g_1(\alpha - 1) + (g_1 - B) - \frac{k_t(1-k_t)}{2} L_0(t)]$$

$$\rho_1(t) = \frac{L_1(t)}{2} k_t(1-k_t) - k_t g_1(\alpha - 1)\beta_t$$

Further, define

$$K_1(t) = K_1^Y(t) + k_t g_1(\alpha - 1)$$

Then

$$\dot{\beta}_t + K_1^T(t)\beta_t + \frac{1}{2}\beta_t^T H_1(t)\beta_t - \frac{L_1(t)}{2} k_t(1-k_t) = 0,\ \beta(T) = 0 \tag{74}$$

$$\dot{\alpha}_t + K_0(t)\beta_t + \frac{1}{2}\beta_t^T H_0(t)\beta_t - \rho_0(t) = 0,\ \alpha(T) = 0 \tag{75}$$

The solution for BSDE:

$$Z_t = \beta_t^T \sigma(Y_t, t)$$

The portfolio holding:

$$\Psi_t = k_t(\sigma_t^R \sigma_t^{R'})^{-1} \mu_t^R + (\sigma_t^{R'})^{-1} Z_t'$$

In order to obtain $Z_t$, we need to solve for (**??**) for $J_t$. To obtain analytical solution, we further assume the pricing kernel is affine. Specifically, in this paper, we assume constant risk-free interest rate $r$, and $\eta_t \cdot \eta_t = L_0(t) + L_1(t)Y_t$, where $Y_t$ is the state variable process

$$dY_t = (K_0(t) + K_1(t)Y_t)dt + \sigma(Y_t, t)dW_t$$

with $\sigma(Y_t,t)\sigma(Y_t,t)^T = L_0(t) + L_1(t)Y_t$

$\beta_t$ solves a Racatti ODE, and the hedge demand for volatility is

$$Z_t = \beta_t^T \sigma(Y_t,t)$$

# B Example: Heston Stochastic Volatility Model

Pricing kernal $\phi_t$

$$\begin{aligned}
\frac{d\phi_t}{\phi_t} &= -[rdt + \eta_1\sqrt{V_t}dB_t^1 + \eta_2\sqrt{V_t}dB_t^2] \\
d\ln S_t &= (r + \eta_1 V_t - \frac{1}{2}V_t)dt + \sqrt{V_t}dB_t^1 \\
dV_t &= \kappa(\bar{V} - V_t)dt + \sigma_V\sqrt{V_t}(\rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2)
\end{aligned}$$

The affine structure has one dimension with $Y_1(t) = V_t$.

$$\eta_t = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \sqrt{Y_1} \Rightarrow \eta_t \cdot \eta_t = (\eta_1^2 + \eta_2^2)Y_1$$

Hence

$$L_0(t) = 0,\ L_1(t) = \eta_1^2 + \eta_2^2$$

$$\begin{aligned}
\rho_0(t) &= -[(1-k_t)(B-r) + \alpha_t k_t g_1(\alpha - 1) + (g_1 - B)] \\
\rho_1(t) &= (1-k_t)k_t\frac{\eta_1^2 + \eta_2^2}{2} - g_1(\alpha-1)k_t\beta_t
\end{aligned}$$

$$\sigma_t = (\rho \quad \sqrt{1-\rho^2})\sigma_V\sqrt{Y_1}$$

We have:

$$\sigma_t\sigma_t^T = \sigma_V^2 Y_1$$

$$\sigma_t\eta_t = \sigma_V(\rho\eta_1 + \eta_2\sqrt{1-\rho^2})Y_1$$

$$K_1^Y(t) = -\kappa - (1-k_t)\sigma_V(\rho\eta_1 + \eta_2\sqrt{1-\rho^2})$$

$$K_1(t) = -\kappa - (1-k_t)\sigma_V(\rho\eta_1 + \eta_2\sqrt{1-\rho^2}) + k_t g_1(\alpha - 1)$$

For $t < T$:

$$\dot{\beta}_t + K_1(t)\beta_t + \frac{1}{2}\sigma_V^2\beta_t^2 - \frac{\eta_1^2 + \eta_2^2}{2}k_t(1-k_t) = 0,\ \beta(T) = 0 \tag{76}$$

$$\dot{\alpha}_t + \kappa\bar{V}\beta_t + [(1-k_t)(B-r) + \alpha_t k_t g_1(\alpha - 1) + (g_1 - B)] = 0,\ \alpha(T) = 0 \tag{77}$$

Where $k_t$ solves:

$$g_1(1-\alpha)k_t^2 - g_1 k_t - \dot{k}_t = 0,\ k_T = 1$$

## B.1 Portfolio

The hedging portfolio:

$$\begin{aligned} \frac{dS_t}{S_t} &= (r + \eta_1 V_t)dt + \sqrt{V_t}dB_t \\ \frac{dg_t}{g_t} &= (r + \sigma_V\rho\frac{g_V}{g}\eta_1 V_t + \sigma_V\sqrt{1-\rho^2}\frac{g_V}{g}\eta_2 V_t)dt + \sigma_V\frac{g_V}{g}\sqrt{V_t}(\rho dB_t + \sqrt{1-\rho^2}dZ_t) \end{aligned}$$

with

$$\mu_t^R = \begin{pmatrix} \eta_1 V_t \\ \frac{g_V}{g}\sigma_V V_t(\rho\eta_1 + \sqrt{1-\rho^2}\eta_2) \end{pmatrix}$$

and

$$\sigma_t^R = \begin{pmatrix} V_t & 0 \\ \sigma_V\frac{g_V}{g}\sqrt{V_t}\rho & \sigma_V\frac{g_V}{g}\sqrt{V_t}\sqrt{1-\rho^2} \end{pmatrix}$$

The portfolio:

$$\begin{aligned} \Psi_t^S &= k_t(\eta_1 - \frac{\rho\eta_2}{\sqrt{1-\rho^2}}) \\ \Psi_t^V &= (k_t\frac{\eta_2}{\sigma_V\sqrt{1-\rho^2}} + \beta_t)\frac{g}{g_V} \end{aligned}$$

The volatility holding can also be decomposed to myopic and hedging components:

$$\begin{aligned} \Psi_{myopic}^V &= (k_t\frac{\eta_2}{\sigma_V\sqrt{1-\rho^2}})\frac{g}{g_V} \\ \Psi_{hedge}^V &= \beta_t\frac{g}{g_V} \end{aligned}$$

The relative importance of demand for volatility between hedge portfolio and myopic portfolio:

$$\frac{\Psi^V_{hedge}}{\Psi^V_{myopic}} = \frac{\beta_t \sigma_V \sqrt{1-\rho^2}}{k_t \eta_2}$$

# C Numerical Scheme for General Recursive Utility

Denote $\beta(n) = \beta(T - n\Delta t)$. Let

$$c_1 = -\kappa - (1 - \frac{1}{\gamma})\sigma_V(\rho\eta_1 + \eta_2\sqrt{1-\rho^2})$$

and

$$c_2 = \frac{1-\gamma}{2\gamma^2}(\eta_1^2 + \eta_2^2)$$

then for

$$|q| \cdot n\Delta t \leq \frac{1}{2}$$

$$\beta(n) = \frac{1}{1+\frac{1}{n}}[\beta(n-1) \cdot (1 + (c_1 - \frac{1}{2}q)\Delta t) + \frac{1}{2}\sigma_V^2 \beta(n-1)^2 \Delta t + c_2 \Delta t]$$

Otherwise, for $|q| \cdot n\Delta t > \frac{1}{2}$

$$\beta(n) = \beta(n-1) + \beta(n-1) \cdot (c_1 - g_1(n\Delta t))\Delta t + \frac{1}{2}\sigma_V^2 \beta(n-1)^2 \Delta t + c_2 \Delta t$$

Final condition being

$$\beta(0) = 0$$

# D Exact Solution to Additive Utility

For comparison purpose, we give the following proposition on the additive utility:

*Under additive utility, i.e., when $\psi = \frac{1}{\gamma}$, for $\gamma \neq 1$, the optimal volatility holding can be decomposed to myopic and hedging components:*

$$\begin{aligned} \Psi^V_{myopic} &= (\frac{\eta_2}{\gamma\sigma_V\sqrt{1-\rho^2}})\frac{g}{g_V} \\ \Psi^V_{hedge} &= \beta_t \frac{g}{g_V} \end{aligned}$$

*where*

$$\beta_t \equiv \frac{h_y(t,Y_t)}{h(t,Y_t)} = \frac{\int_t^T e^{G(s-t)+H(s-t)Y_t} H(s-t)ds}{\int_t^T e^{G(s-t)+H(s-t)Y_t} ds} \tag{78}$$

*where*

$$h(t,Y_t) = \frac{1}{1-\gamma}\int_t^T e^{p(s-t,Y_t)}ds$$

*with* $p(\tau,y) = G_\tau + H_\tau y$*, and* $k = \frac{1}{\gamma}$*, and* $G_\tau$ *and* $H_\tau$ *are defined as*

$$\begin{aligned} H_\tau &= \frac{\exp(K_2\tau)-1}{2K_2+(K_1+K_2)(\exp(K_2\tau)-1)}\delta \\ G_\tau &= \frac{2\kappa\bar{V}}{\sigma_V^2}\ln\left(\frac{2K_2\exp((K_1+K_2)\tau/2)}{2K_2+(K_1+K_2)(\exp(K_2\tau)-1)}\right) + \frac{1-\gamma}{\gamma}\left(r-\frac{\beta}{1-\gamma}\right)\tau \end{aligned}$$

*with*

$$\begin{aligned} K_1 &= \kappa + \left(1-\frac{1}{\gamma}\right)\sigma_V\left(\rho\eta_1+\eta_2\sqrt{1-\rho^2}\right) \\ \delta &= \frac{1-\gamma}{\gamma^2}(\eta_1^2+\eta_2^2) \\ K_2 &= \sqrt{K_1^2-\delta\sigma_V^2} \end{aligned}$$

The relative importance of demand for volatility between hedge portfolio and myopic portfolio:

$$d_t \equiv \frac{\Psi^V_{hedge}}{\Psi^V_{myopic}} = \frac{\beta_t\sigma_V\sqrt{1-\rho^2}}{k_t\eta_2}$$