Weak Interest Rate Parity and Currency Portfolio Diversification

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ABSTRACT

This paper presents a dynamic model of optimal currency returns with a hidden Markov regime switching process. We postulate a weak form of interest rate parity that the hedged risk premiums on currency investments are identical within each regime across all currencies. Both the in-sample and the out-of-sample data during January 2002 - March 2005 strongly support this hypothesis. Observing past asset returns, investors infer the prevailing regime of the economy and determine the most likely future direction to facilitate portfolio decisions. Using standard mean variance analysis, we find that an optimal portfolio resembles the Federal Exchange Rate Index which characterizes the strength of the U.S. dollar against world major currencies. The similarity provides a strong implication that our three-regime switching model is appropriate for modeling the hedged returns in excess of the U.S. risk free interest rate. To investigate the impact of the equity market performance on changes of exchange rates, we include the S&P500 index return as an exogenous factor for parameter estimation.

1. Introduction

One of the key trends to emerge in recent years has been the growing investment interest in non-traditional asset classes, not only as stand-alone investment instruments but also as value-added components of an asset allocation program. Among possible alternative instruments is a well formulated multi currency portfolio. When determining the building blocks of asset allocation, investors will typically first review the split between equities, fixed income instruments and cash. Recent exchange rate rises and falls against the U.S. dollar indicate that portfolio managers denominated in the U.S. may need to carefully consider their currency overlays in the international investment arena.

Portfolio theory has been applied successfully in a variety of situations in which investments are comprised of various assets from different industry sectors. The assumption that asset returns are normally distributed is key for the theory. However, empirical evidence suggests that asset returns may not be stationary in time, exhibiting fat tails and jumps; see, e.g. Erb et al. (1994), King et al. (1994), Longin and Solnik (1995, 2001), and De Santis and Gerard (1997). The time dependence of returns has been modeled in a number of ways. Xia (2001) provided a framework of dynamic learning for portfolio optimization with two assets. Ang and Bekaert (2002) present an international asset allocation model with regime switching.

One particular feature of the time variation in returns on assets is the asymmetric correlation phenomenon, where the relationship between asset returns changes for down and up markets. Our first goal is to formulate a model for currency returns that reproduces the asymmetric correlation phenomenon in terms of Markov regime switching. Related to asymmetric correlations is the premium on foreign exchange rates. Market conditions suggest that forward exchange rates should be unbiased predictors of future spot rates and there is no risk premium. However, the condition does not hold up in reality. Bansal (1997) provided an explanation for the violation of uncovered interest rate parity and suggested a particular term structure model accounting for the puzzling empirical evidence. Bekaert and Hodrick (1993) proposed a regime switching model for measurement of foreign exchange risk premiums.

What could be a reason for the interest rate parity to be violated in practice? It may depend on how risk and return are defined. Different measurements may end up with different conclusions. We look at the covered interest rate parity using a continuous series of futures contracts. Empirical evidence indicates that hedged excess returns on all major currency investments exhibit no risk premium using statistical tests based on normality. However, the distributions of the hedged excess returns are highly skewed and excess kurtosis is too large; see Baz et al. (2001). We hypothesize that the covered interest rate parity does not hold in its strong sense but in a weak version. We assume that there are regimes and that the expected returns on the hedged currency investments are constant across all currencies within each regime. This is the result of the nominal rates being almost constant for each currency and each regime of the economy; see Gray (1996). We term this assumption the *weak interest rate parity condition*. Supporting evidence and tests of this hypothesis are provided in later sections.

Based on the existence of regimes in the foreign exchange market, an optimal currency portfolio model is formulated. There has been a resurgence of interest in dynamic asset allocation where investment opportunity sets change over time. Research shows that a Markov switching model can explain the skewness and excess kurtosis. In most of the literature, time varying parameters are captured by a linear function of the state variables. In contrast, expected returns and volatilities may vary with regime switching, rather than with an explicitly specified function relation of the state variables, see Hamilton (1989), Gray (1996), and Kim (1994). In our benchmark model, information on state variables is implicitly incorporated in determining returns on currency investments. To capture market expectation, we include the S&P500 index returns in the estimation of the parameters for currency returns. The reason for this inclusion is that returns on equity markets are statistically correlated with returns on the currency markets; see, e.g. Roll (1992) and Warren and Chung (1995). For empirical analysis, a three regime model is specified with the restriction that the weak interest rate parity condition holds. Due to the special features of our model, the estimation is more computationally involved than a standard algorithm for estimating a Markov switching model.

The regime switching model and the performance of the equilibrium portfolios are contrasted with the Federal Exchange Rate Index (FERI) and the U.S. treasury bill. The FERI index characterizes the strength of the U.S. dollar against world major currencies. We expect to see that a hedged portfolio using futures contracts has a similar performance to that of the treasury bill over time, while an unhedged portfolio has a similar performance to that of the FERI index in equilibrium. The similar performances of both the hedged and unhedged portfolios to those of the treasury bill and the FERI index, respectively, indicate that our three regime model is appropriately specified, with inclusion of the S&P 500 index as exogenous information for parameter estimation. Hence, portfolios including the S&P 500 index for the model estimation should perform better with both in-sample and out-of-sample.

Section 2 presents the setting of the model and the weak interest rate parity hypothesis in the framework of Markov switching. Section 3 discusses parameter estimation using a modified EM algorithm to accommodate the constraints due to the weak interest rate parity condition. We further elaborate the importance of incorporating market economic factors into the estimation procedure. Section 4 discusses portfolio performance with both in-sample and out-of-sample data. Section 5 concludes.

2. Model Setting and Formulation

2.1. Excess Returns in Foreign Currency Markets

The basis for considering regimes in foreign exchange markets is covered interest rate parity. It is well-known that if the nominal rates are fixed over time, the forward premium must equal the differential of the nominal interest rates between the home and the foreign countries. In our analysis, the home country is the US and the numeraire is the US dollar. To formalize the relationships, let E_{tj} be the dollar price of currency j at time t. Then, the continuously compounded rate of depreciation/appreciation of the dollar against currency j is $\ln(E_{t+1,j}/E_{tj})$. Let r_t and r_{tj} be the continuously compound nominal interest rates of the home country and country j, respectively. With constant nominal rates, $r_t = r$ and $r_{tj} = r_j$, an investor can generate a riskless portfolio using currency futures contracts. Let F_{tj} be the futures price at time t with delivery date T, then the interest rate parity condition (no risk premium) is

$$F_{tj} = E_{tj}\mathrm{e}^{(r-r_j)(T-t)}.$$

Let $f_{tj} = \ln(F_{t+1,j}/F_{tj})$ and $e_{tj} = \ln(E_{t+1,j}/E_{tj})$. A variant of the above parity is

$$e_{tj} - f_{tj} + r_j - r = 0.$$

The validity of the above equation follows from the assumption that the nominal rates are fixed over time. However, in reality the equation never holds exactly true since short rates fluctuate. In other words, even the return on a fully hedged position using futures contracts is subject to risk. The uncovered dollar return on a continuously compounded currency j money market investment is $E_{t+1,j}/E_{tj}\mathrm{e}^{r_j}$. With time varying nominal rates, the excess dollar rate of return on a currency money market investment is

$$r_{tj} - r_t + e_{tj}.$$

Let

$$R_{tj} = e_{tj} - f_{tj} + r_{tj} - r_t. \tag{1}$$

Then R_{tj}, the *hedged excess return* on a currency, is the excess return of an investment in currency j with a fully hedged position in the futures contract. It is expected that R_{tj} has little value in terms of expectation but substantial risk. Our concern is with modeling the risk of R_{tj} and the implication for equilibrium allocation of capital in the currency market. With a fully hedged position, a representative investor should earn a substantial amount while risk is controlled to its minimum.

2.2. Returns as Mixtures of Normals

A particular violation of the parity condition is the situation where market regimes exist, so that switching between regimes causes the violation of the covered interest rate parity. The essential concept in the Markov switching model is that the dependent variables in the system vary by regime. Given a finite number of regimes, the behavior of the dependent variables is determined by which regime the model is in at a point in time. The likelihood of being in a particular regime is determined by a set of transition probabilities.

Assume M_t is a Markov chain, which can take exclusively any one of k states, viewed as regimes of the economy. Suppose that, the initial regime is M_0 with probability $q_i = \Pr[M_0 = i]$, $i = 1, ..., k$. The transition matrix for the Markov chain is

$$P = \begin{bmatrix} p_{11} & \cdots & p_{1k} \\ \vdots & \ddots & \vdots \\ p_{k1} & \cdots & p_{kk} \end{bmatrix}. \tag{2}$$

where

$$p_{ij} = \Pr[M_{t+1} = j \mid M_t = i].$$

The Markov property is that the conditional probability distribution of the process at time s given the whole history of the process up to and including time $t < s$, depends only on the state of the process at time t.

Under the interest rate parity condition, the hedged excess return, R_{tj}, is assumed to carry no risk premium, but empirical evidence suggests that the expected return on a hedged position has different risk premiums in different regimes of the economy; see Evans (2002). To accommodate that evidence, we assume that the hedged risk premiums across all foreign currencies are equal in each regime of the economy to avoid currency arbitrage. This premise is referred to as *weak interest rate parity*.

The implication of weak interest rate parity is that $R_{t1}, ..., R_{tn}$ have the same mean but different variances in each regime. Given that the regime at time t is $M_t = i$, then conditionally $R_{t1}, ..., R_{tn}$ are jointly normally distributed with common mean $\mu_i = E[R_{tj} \mid M(t) = i]$, for $j = 1, 2, ..., n$, and the variance covariance matrix

$$\sigma_i = \begin{bmatrix} \sigma_{11i} & \cdots & \sigma_{1ni} \\ \vdots & \ddots & \vdots \\ \sigma_{n1i} & \cdots & \sigma_{nni} \end{bmatrix}. \tag{3}$$

Assuming the regimes over time are not observable, the unconditional joint distribution of $R_{t1}, ..., R_{tn}$ is a mixture of k n-normals. The mixing coefficients are the transition probabilities depending on the prevailing regime at time t. Given that the prevailing regime is i at a point in time, the one-period conditional expected return is

$$\bar{\boldsymbol{R}}_i = \sum_{j=1}^{k} p_{ij} \boldsymbol{\mu}_j. \tag{4}$$

Since the hedged risk premiums are all equal across assets within each regime, the conditional variance-covariance matrix is,

$$\boldsymbol{V}_i = \sum_{j=1}^{k} p_{ij} \left(\boldsymbol{\sigma}_j + (\boldsymbol{\mu}_j - \bar{\boldsymbol{R}}_i)^2 \boldsymbol{J} \right), \tag{5}$$

where $\boldsymbol{J}$ is the $n \times n$ matrix with all entries equal to 1.

However, the values of all parameters, including q_j, $\boldsymbol{P}$, $\boldsymbol{\mu}_j$ and $\boldsymbol{\sigma}_j$, are unknown to the investors and estimates are required for portfolio decision and implementation. The assumption of the weak interest rate parity poses an element of complexity in parameter estimation. Tsao and Wu (2005) developed a nonparametric method for estimation of a common mean with different variances based on empirical likelihood, and the procedure can be adapted to the Markov switching process.

The estimation procedure becomes even more complex when extra information, such as equity and/or bond returns are considered. Research work has documented that equity market returns are negatively correlated with foreign exchange rates. This information is useful for improving the accuracy of parameter estimates. The extra variables may be introduced through regression based models; see Hamilton (1989). However, it is possible to incorrectly specify a functional relationship. To avoid such biases in parameter estimation, we assume that the impact of economic factors or other relevant market securities are implicit through their correlations.

This setting requires us to study how returns on currency investments are related to changes of asset returns in other categories.

To consider the relationship between currencies and other assets, assume the selected n currencies and the m exogenously specified economic factors are driven by the same Markov chain. Let $Q_{t1}, ..., Q_{tm}$ be the rates of change in the factors. Given $M(t) = i$, the factors and the currencies

$$Q_{t1}, ..., Q_{tm}, R_{t1}, ..., R_{tn}$$

are jointly normally distributed, with mean vector

$$\tilde{\mu}_i = [\alpha_{i1}, ..., \alpha_{im}, \mu_i, ..., \mu_i]$$

and variance covariance matrix

$$\begin{bmatrix} \tau_i & \gamma_i \\ \gamma_i^\top & \sigma_i \end{bmatrix}.$$

τ_i is the $m \times m$ covariance matrix of $Q_{t1}, ...Q_{tm}$ and γ_i the $m \times n$ covariance matrix between $Q_{t1}, ..., Q_{tm}$ and $R_{t1}, ..., R_{tn}$, for $i = 1, ..., k$. This expanded setting poses challenges in parameter estimation. Details on estimation in the expanded model are given in Section 3 and the Appendix.

Another difficulty in applying the Markov switching model is the determination of the regimes over time. At each point, investors should know the state of the economy in determining their portfolio decision. They need to find a way of acquiring such information to facilitate further decision making. If outreach is not available, investors must then estimate the probability of a regime from observation of past data and statistically infer the prevailing regime. This amounts to calculating the posterior probabilities for the regime states at each point in time.

2.3. The Objective of Risk Diversification

With values (estimates) for parameter in the switching model, the optimal portfolio for risk minimization can be considered. In the standard portfolio problem an investor is presented at each time point with a variety of investment opportunities, and a decision is made to allocate available capital to the various assets. At decision points, the investor has information on past prices and preferences for the growth and security of capital. We assume that all investors maximize the expected growth rate of capital. This strategy is equivalent to maximizing the log of accumulated capital, and it is myopic since the objective function is separable and the decision is Markovian; see Hakansson 1971. Under this assumption (with expected growth rate objective), the optimal investment portfolio also maximizes the Sharpe ratio over time. The two fund separation theory states that all investors will invest in the same risky portfolio (constructed by risky assets only) but with different proportions of their total wealth in the risky asset portfolio. So, the problem reduces to a sequential problem of mean variance optimization by maximizing the Sharpe ratio over time.

We have proposed a switching model for characterizing hedged excess returns over time. Is there any evidence in portfolio performance to support regime switching? If sample means and sample variances and covariances within regimes can be used as parameter estimates, then portfolio theory says that the equilibrium portfolio should

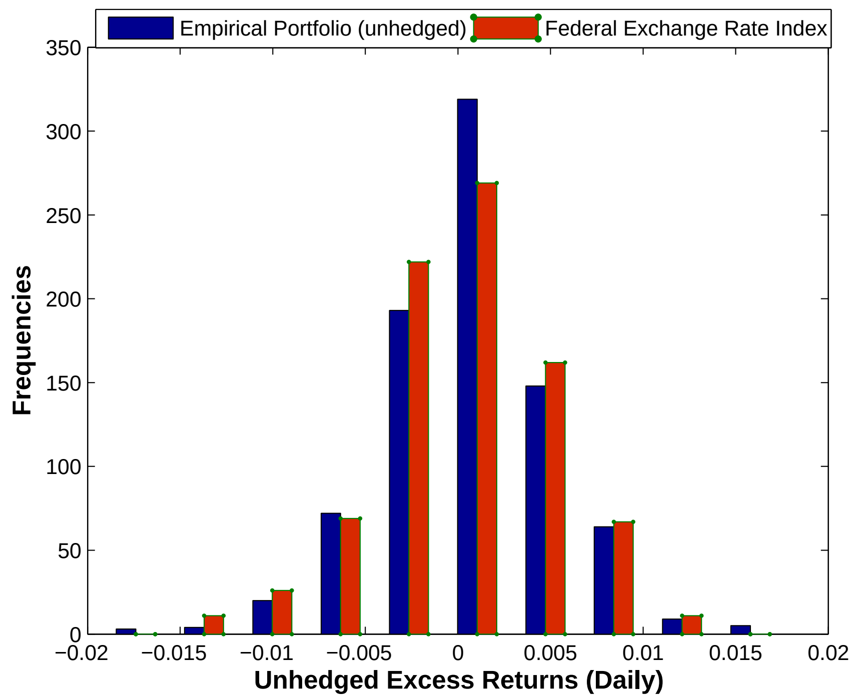

Figure 1. unhedged Excess Currency Returns in Equilibrium

have performance which is close to a broadly diversified benchmark. Figure 1 provides an example of why we should look for other modeling approaches than constant interest rates (interest rate parity) for the dynamics of exchange rates. The return on an unhedged portfolio does not match the return in the FERI.

With the regime switching model, let $w_{ti} = \begin{bmatrix} w_{t1i}, & \cdots, & w_{tni} \end{bmatrix}^\top$ be the weight vector in the n foreign currencies in regime i and at time t. Then $1 - \sum_{j=1}^{n}(w_{tji})$ is the proportion allocated to the home currency (the U.S. dollar), the numeraire for our model. For $t = 1, ..., T$ and $i = 1, ..., k$, the holding period return of the portfolio is

$$r_t + \sum_{j=1}^{n} w_{tji} R_{tji},$$

where R_{tji} is the hedged excess return of currency j in regime i and at time t as defined in (1). For regime i, the expected excess return of the hedged portfolio is

$$\bar{R}_{ti} = \bar{R}_i \sum_{j=1}^{n} w_{tji}, \qquad \forall i = 1, ..., k.$$

and its variance covariance matrix is

$$\Omega_{ti} = w_{ti}^{\top} V_i w_{ti}.$$

Since Merton (1973) shows that maximizing the expected growth of the portfolio is instantaneously mean variance efficient, investors generate their portfolios by maximizing the risk-adjusted expected return

$$\max_{w_{ti}} \left\{ r_t + \bar{R}_{ti} - \tfrac{1}{2}\lambda\Omega_{ti} \right\}, \tag{6}$$

where λ is investor's risk aversion parameter. Assuming the market is free of friction, the solution to (6) is standard, and it can be obtained analytically by calculus. The optimal weights are

$$w_{ti} = \frac{\bar{R}_i}{\lambda} V_i^{-1}\mathbf{1}, \qquad \forall i = 1, ..., k, \tag{7}$$

where $\mathbf{1}$ is the unit vector of length n. Equation (7) implies that all investors allocate fixed proportions to foreign currencies, though different investors may have different amount in all currencies. Hence, in equilibrium, all investors will share the same foreign currency portfolio which is

$$\bar{w}_{ti} = \frac{V_i^{-1}\mathbf{1}}{\mathbf{1}^{\top} V_i^{-1}\mathbf{1}}. \tag{8}$$

That is, investors portfolio weights among foreign currencies depend on the prevailing regime and its risk level characterized by the conditional variances. In equilibrium, the market portfolio is identical to the equilibrium portfolio since all investors hold the same foreign currency portfolio (8). The optimal foreign currency portfolio is the minimum variance portfolio. The intuition is that, when the risky assets have a common risk premium, the minimum variance portfolio coincides with the tangency portfolio. Hence, the efficient frontier is a single point.

How does an individual investor optimally allocate his/her capital over time? Depending on the risk/return profile (or a target return level), they will allocate the total capital between the home currency and the optimal foreign currency portfolio. To be specific, by (6) investors with risk aversion λ will invest $\frac{\bar{R}_i}{\lambda}\mathbf{1}^\top V_i^{-1}\mathbf{1}$ percent of the total wealth in the optimal currency portfolio over time. The rest of wealth is invested in the home currency.

Based on Markowitz (1952) mean variance analysis and the Capital Asset Pricing Model developed by Sharpe (1964), all investors will hold the same risky portfolio in equilibrium. This risky portfolio is generated by maximizing the Sharpe ratio. Since the expected returns on hedged currency investments are assumed to be constant across all currencies, the tangency portfolio and the minimum variance portfolio converge to the same point on the efficient frontier. That is, the investors' objective is equivalent to minimizing risk among all currencies. We examine whether the equilibrium theory under mean variance analysis will hold in the currency market. We use the Federal Exchange Rate Index that characterizes the strength of the U.S. dollar against world major currencies as a proxy for the market return on currency investments. With regime switching and weak interest rate parity, we expect to see that the optimally hedged portfolio has a similar performance to that of the U.S. dollar, while the unhedged optimal currency portfolio has a similar performance to the currency index in equilibrium. We examine whether this expectation holds in Section 4.

3. Model Estimation

Having derived the optimal portfolio in (8), we now turn to the procedure for parameter estimation. Small estimation error of parameters can greatly change the formalism of the optimal portfolio; see e.g. Kallberg and Ziemba (1984), Best and Grauer (1991), and Chopra and Ziemba (1993). To enhance the estimation accuracy, we propose a new estimation procedure. The algorithm is a modified EM algorithm that is tailored to suit our restrictive model.

3.1. The Algorithm

Given the special structure of the switching model for currency returns, it is necessary to develop an algorithm for solving for the maximum likelihood estimates of parameters in the model. With Markov switching models, the estimation is an adaptation of the EM algorithm which consists of two steps, the E-Step and the M-Step. Given an initial condition, the two steps alternate in updating parameters. This iteration algorithm guarantees to converge to a local optimal point.

Let Θ be the set of all parameters, x_t the observations on hedged currency returns and asset returns, and y_t the unobservable regimes. Let $x = (x_1, ..., x_T)$ and $y = (y_1, ..., y_T)$. The full maximum likelihood estimator is calculated from the log of the marginal likelihood.

$$\max_{\Theta} \quad \ln P(x;\Theta) = \ln \sum_{y\in\mathcal{Y}} P(x,y;\Theta) = \ln \sum_{y\in\mathcal{Y}} P(x|y,\Theta)P(y;\Theta)$$

where y takes values in the sample space $\mathcal{Y}$ of the regimes.

This problem is difficult to solve due to the summation inside the log function. However, we can use a sequential iteration method to reduce the computational complexity. The following inequality holds,

$$\ln P(x;\Theta) \geq \sup_f \left\{ \sum_{y\in\mathcal{Y}} f(y) \ln \frac{P(x,y;\Theta)}{f(y)} \right\}$$

where $f(\cdot) \in \mathcal{F}$, the set of probability distribution functions on $\mathcal{Y}$. The right hand side is a lower bound for the log likelihood of the parameter Θ. The objective is to maximize the lower bounds (to find a distribution) such that the bound is tight for the given data and the parameters. Then, the parameter Θ is updated by maximizing the expected log likelihood with the obtained probability measure.

The inner maximization problem is solved by calculus of variation to obtain

$$f(y) = P(y|x;\Theta).$$

This optimality can be written as

$$\sup_{\Theta} \left\{ \sup_{Q} \left\{ E^Q \left[\ln P(x|y;\Theta) \right] \right\} \right\},$$

where Q is an arbitrary probability distribution on $\mathcal{Y}$. Equivalently, we look for a probability measure Q and the related parameters so that the likelihood is maximized. The algorithm proceeds as follows:

(1) Set an initial value for $\Theta^{(0)}$.

(2) Find the prior distribution of y for the given $\Theta^{(0)}$. Then calculate the conditional distribution given the data, $P(y|x;\Theta^{(0)})$.

(3) Maximize the expected log likelihood with the probability distribution function on y from [2], $P(y|x;\Theta^{(0)})$. That is

$$\max_{\Theta^{(1)}} \sum_{y\in\mathcal{Y}} P(y|x;\Theta^{(0)}) \ \ln P(x|y;\Theta^{(1)}).$$

(4) With $\Theta^{(1)}$ as the new initial value, return to [2]. The process continues until some stopping criterion is satisfied.

The details of the algorithm are in Appendix A. Updating with the normal distribution is straightforward, since one needs only calculate the sample means and the sample variances. However, our problem is constrained, so that the means across all currencies are equal within each regime with possible unequal means for different regimes. The following proposition is useful in implementing the algorithm.

Proposition 1. *Let $x_1, ..., x_S$ be independently drawn from a multivariate normal distribution with equal mean value μ, and the variance covariance matrix, σ. The optimal solution, $(\hat{\mu}, \hat{\sigma})$, to the following maximization problem*

$$\max_{\mu,\sigma} \sum_{t=1}^{S} p_t \ln \phi(x_t, \mu\mathbf{1}, \sigma)$$

satisfies

$$\begin{cases} \hat{\mu} &= \dfrac{\sum_{t=1}^{S} p_t x_t^\top \hat{\sigma}^{-1}\mathbf{1}}{\mathbf{1}^\top \hat{\sigma}^{-1}\mathbf{1}} \\ \hat{\sigma} &= \sum_{t=1}^{S} p_t (x_t - \hat{\mu}\mathbf{1})(x_t - \hat{\mu}\mathbf{1})^\top. \end{cases}$$

where $p_1, ..., p_S$ are nonnegative numbers with a sum equal to 1 and ϕ is a multivariate normal density function with n components.

Since $\hat{\mu}$ is a scalar, we can use Newton's search method to find the solution for $\hat{\mu}$ and then calculate $\hat{\sigma}$.

Another complication in the estimation process arises from introducing exogenous variables. Let $z_1, ..., z_S$ be independently drawn from a multivariate normal distribution with m components. Assume that $x_i' s$ and $z_j' s$ are jointly normally distributed with mean vector partitioned as

$$[\alpha_1, ..., \alpha_m, \mu, ..., \mu]^\top =: \begin{bmatrix} \alpha \\ \mu \mathbf{1} \end{bmatrix}$$

and the variance-covariance matrix

$$\Omega = \begin{bmatrix} \gamma & \delta \\ \delta^\top & \sigma \end{bmatrix}.$$

Then the maximum log likelihood estimates can be represented in the form of simultaneous equations. Proposition 2 develops the solution procedure.

Proposition 2. *Let $x_1, ..., x_S$ and $z_1, ...z_S$ be given as above. The optimal solution, $(\hat{\mu}, \hat{\alpha}, \hat{\Omega})$, to the following maximization problem*

$$\max_{\mu, \alpha, \Omega} \quad \sum_{t=1}^{S} p_t \ln \phi \left((z_t, x_t), [\alpha, \ \mu \mathbf{1}], \Omega \right),$$

satisfies

$$\begin{cases} \hat{\mu} &= \dfrac{\sum_{t=1}^{S} p_t(\boldsymbol{x}_t^\top \bar{\Omega}_{22} + \boldsymbol{z}_t - \hat{\boldsymbol{\alpha}}^\top \bar{\Omega}_{12}))\boldsymbol{1}}{\boldsymbol{1}^\top \bar{\Omega}_{22}\boldsymbol{1}} \\ \hat{\boldsymbol{\alpha}} &= \sum_{t=1}^{S} p_t(\boldsymbol{z}_t - \bar{\Omega}_{11}^{-1}\bar{\Omega}_{12}(\boldsymbol{x}_t - \hat{\mu}\boldsymbol{1})) \\ \hat{\Omega} &= \sum_{t=1}^{S} p_t \Omega_t. \end{cases}$$

$p_1, ..., p_T$ *are nonnegative numbers with a sum equal to* 1 *and*

$$\Omega_t = \begin{bmatrix} (\boldsymbol{z}_t - \hat{\boldsymbol{\alpha}})(\boldsymbol{z}_t - \hat{\boldsymbol{\alpha}})^\top & (\boldsymbol{z}_t - \hat{\boldsymbol{\alpha}})(\boldsymbol{x}_t - \hat{\mu}\boldsymbol{1})^\top \\ (\boldsymbol{x}_t - \hat{\mu}\boldsymbol{1})(\boldsymbol{z}_t - \hat{\boldsymbol{\alpha}})^\top & (\boldsymbol{x}_t - \hat{\mu}\boldsymbol{1})(\boldsymbol{x}_t - \hat{\mu}\boldsymbol{1})^\top \end{bmatrix},$$

with

$$\hat{\Omega}^{-1} = \begin{bmatrix} \bar{\Omega}_{11} & \bar{\Omega}_{12} \\ \bar{\Omega}_{21} & \bar{\Omega}_{22} \end{bmatrix}.$$

Substituting $\hat{\Omega}$ in the expressions for $\hat{\mu}$ and $\hat{\boldsymbol{\alpha}}$ reduces the solution to an iteration procedure of $m + 1$ variables[1].

4. Empirical Analysis

The presence of regimes and the weak interest rate parity condition is now tested with data on major currencies. The regimes will be validated based on the normality of hedged currency returns. The weak interest rate parity hypothesis is tested using analysis of variance and pairwise comparisons of means within regimes.

In addition to the U.S. dollar, five major currencies are used in our study: the Australian dollar (AUD), the Canadian dollar (CAD), the euro (EURO), the Japanese

[1]Matlab codes are available on request.

yen (JPY), and the British pound (GBP). Their daily exchange rates to the U.S. dollar during January 2002 - March 2005 were collected from the Datastream database. The nominal rates are the short interest rates (available at the Datastream database). Time series of futures prices are generated as the average futures prices available on each day. By formula (1), we first calculate the hedged excess returns, R_{tj}, for money market j. The accumulated hedged excess returns are plotted in Figure 2

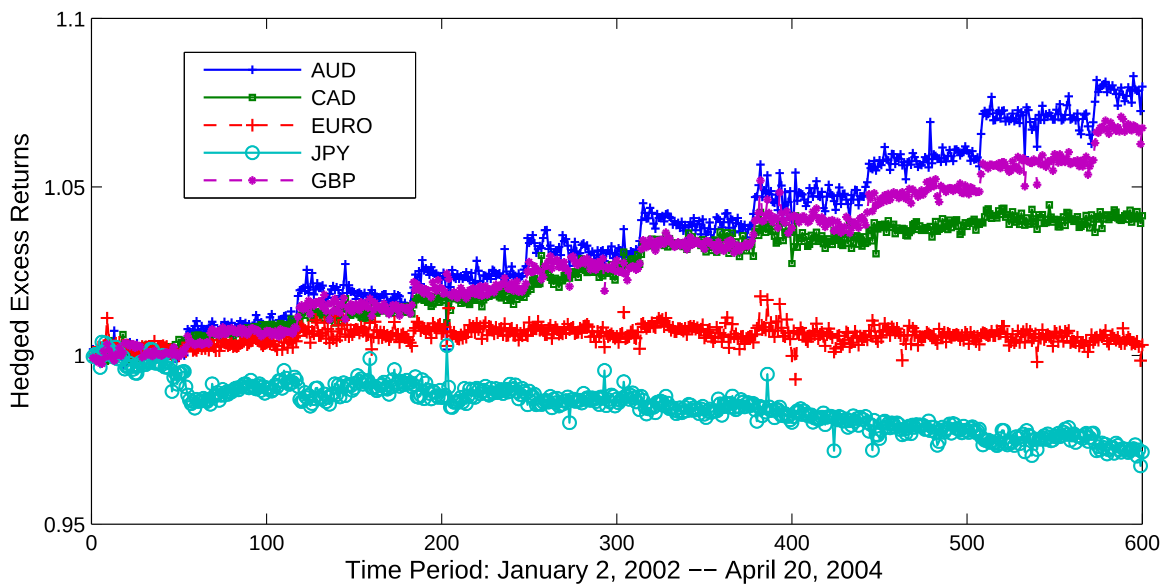

Figure 2. Except for the Japanese yen and the euro, all other currencies have strong performance against the dollar. The strong performances of the Australian dollar and the Canadian dollar are due to the effect of large standard deviations and correlations in the market. The sample means are not efficient or not unbiased estimators of the actual means, since the normality is violated and the distributions over time are not homogeneous.

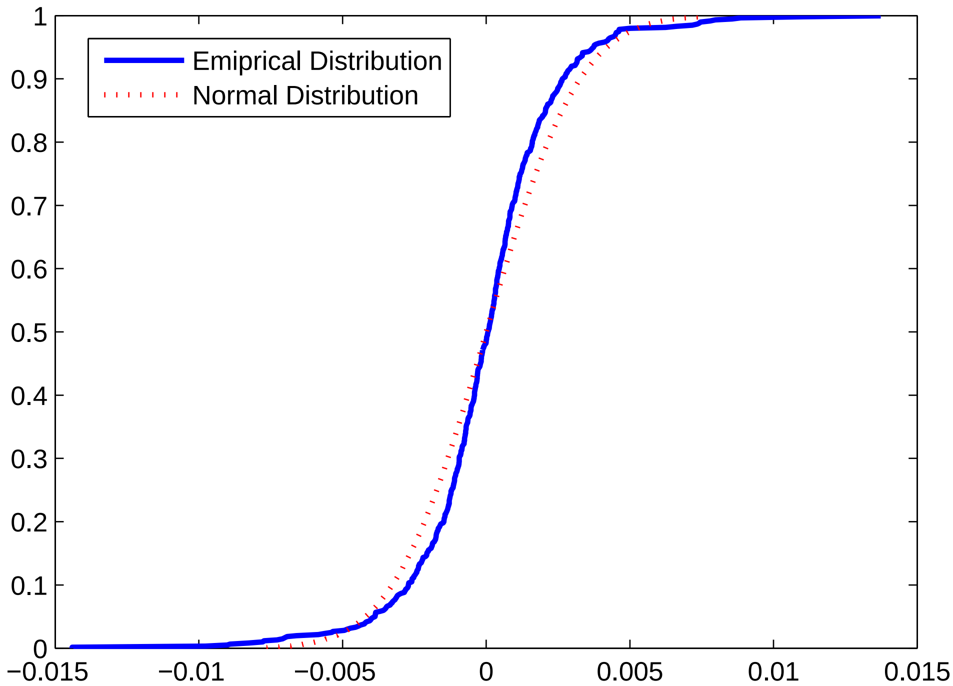

Figure 3. Non-Normality of Daily Returns for EURO.

4.1. Tests for Regimes

We assume that the unconditional distribution of R_{tj} is a mixture of normals, violating the normality assumption for the underlying data generating process. To illustrate the non-normality, consider Figure 3 which depicts the cumulative distribution of returns during January 2, 2002, – April 20, 2004.

We performed Lilliefors and Jarque-Bera tests with the observed time series to verify the validity of the assumption. Both tests are 2-sided test of normality with sample mean and sample variance used as estimates of the population mean and variance, respectively. The sample descriptive statistics and the statistical test results are given in Table I.

Both the Jarque Bera and Lilliefors tests suggest that the null hypotheses be rejected in favor of non-normality for all hedged currency returns. The t-statistic shows that the means of $\boldsymbol{R}_{tj}$ for all currencies are consistent with the null hypothesis. However, the t-tests are performed on the assumption of normality, and the results are questionable.

With the rejection of normality, the presence of regimes was considered. The regimes are the basis of the weak interest rate parity condition. All currencies have an equal hedged risk premium in each regime. The first 600 data points from the data set are used for model training to obtain the estimates of the parameters. A Markov process with 3 regimes is suitable for characterizing the economic uncertainty in the currency market. The 3 regimes can be interpreted as the economic stages of recession, normalization, and expansion, as in Hamilton (1989). Tables II and III present the estimates of the initial regime probability $\boldsymbol{q}$, the transition matrix $\boldsymbol{P}$, the expected excess returns $\boldsymbol{\mu}$ and the volatility matrices $\boldsymbol{\sigma}$. To initialize the algorithm, we use a random vector and a stochastic matrix for the vector $\boldsymbol{q}$ and the transition matrix $\boldsymbol{P}$, respectively. For the hedged risk premium $\boldsymbol{\mu}$ and the volatility matrix $\boldsymbol{\sigma}$, we divide the sample into three groups randomly and use the sample mean and the covariance matrix of each group.

To validate the proposed 3-regime switching model, we perform the Lilliefors and the Jarque-Bera tests for each regime to see whether they are normally distributed. Based on the posterior probabilities, three subsamples are generated using the Viterbi algorithm. For the three regimes, there are 92, 34, and 474 data points, respectively, for the case of excluding the S&P 500. For the case of including the S&P500, there are 365, 101, and 134, respectively. If a permutation is made by the size of risk premiums, the corresponding order of inferred regimes for the second approach should be 101, 134, and 365, which implies a 66% of regimes in common for the two approaches.

The null hypothesis that the conditional marginal distribution for each currency is normally distributed for each regime cannot be rejected by neither Lilliefors nor Jargue Bera tests. That is, the hedged risk premium for each currency has a conditional normal distribution for each regime.

4.2. Impact of Equity Returns on Parameter Estimation

Standard approaches for estimating means and variances are based on the sample statistics of the random variables. Auxiliary variables are rarely considered in practice, even if they are correlated to study variables. Exclusion of such variables may lead to inaccurate estimates. The equity market is negatively correlated with the currency market. We include such information in our estimation procedure. From a U.S. investor's point of view, we consider returns on the S&P500 index for our parameter estimation in the framework discussed in Section 2.

From the estimation results, we observe some dramatic changes when the S&P500 is included as an exogenous factor for the estimation. Since estimation results may depend on initialization, we set the initial values to be equal for the two estimation approaches to make a fair comparison. First, the hedged risk premiums in the three regimes are changed from $(3.2216, -1.8641, 0.2086)$ to $(0.2480, 0.8247, 0.1826)$ in base points. Furthermore, changes in standard deviation and correlation terms are also obvious. Overall, risk levels measured by variances and covariances are dropped substantially, indicating a strong effect from including the index in the estimation procedure. As well, all other parameters such as the transition probability matrix are different using the different estimation techniques; see Tables II and III.

Due to an identification problem of regimes, the estimated regime sequences from the two approaches may be different. In order to find the percentage of regimes that are in common by the two estimation approaches, we need to set a criterion for classifying regimes. If we consider regimes in increasing order of the risk premiums, then regimes 1, 2, and 3 for excluding the S&P500 correspond to regimes 2, 3, and 1 for including the S&P500. In this way of classifying regimes, the two approaches imply a 66% of coincidence in the two inferred regime sequences, which is incredibly high.

4.3. Weak Interest Rate Parity Tests

The back testing shows that our specification of a three regime model is appropriate, with normality within regimes. A t-test is performed for each pair of currencies to see whether the hypothesis of the weak interest rate parity holds within each regime. Both techniques passed all the tests with high p-values; see Tables IV and V.

We also present an F-test for the weak interest rate parity. Although a paired sample test for equal means does not need the assumption of equal variances (as long as the two samples are of the same length and independent normally distributed), a joint test maybe even more powerful asymptotically. We conduct an F-test for equal means, ignoring the assumption of equal variances. The results are in Tables X and XI. Consistent with the pairwise tests, the F-tests show strong support for the weak interest rate parity hypothesis.

4.4. Optimal Portfolio Weights and Capital Accumulation

Having estimated the parameters required for the optimization model, we now study the portfolio performance and obtain the equilibrium implication. To examine the value of information we also study a steady state portfolio which is generated using the steady state probability of the Markov chain to find the unconditional distribution. The steady state probability indicates how often the chain will stay in that state (regime). Both the estimated models with or without the S&P 500 index imply that there exists a steady state distribution for each of the model. The steady state probability is

$$\lim_{n\to\infty} \quad [1, 0, ..., 0] \cdot \boldsymbol{P}^n.$$

For the model excluding the S&P500 index, the steady state probability distribution is $q_{s1} = \begin{bmatrix} 0.2604 & 0.5958 & 0.1438 \end{bmatrix}$. The probabilities are $q_{s2} = \begin{bmatrix} 0.1042 & 0.7143 & 0.1815 \end{bmatrix}$ for the case including the index.

The optimal portfolio weights in each regime are given in Table IV. We see that our currency portfolio is able to capture the momentum of the growth of foreign currencies. In particular, the Canadian dollar did extremely well. The currency had an annualized 8.83% rate of appreciation against the U.S. dollar with a standard deviation of 8.14% from January 2002 to April 2004. Figure 4 depicts the hedged and unhedged optimal portfolio performances for the two estimation approaches. The implication of auxiliary information on regimes is important as evidenced from differences in the transition probabilities and the estimated excess risk premiums and covariances. Also, the implied regimes are distributed differently for the two approaches.

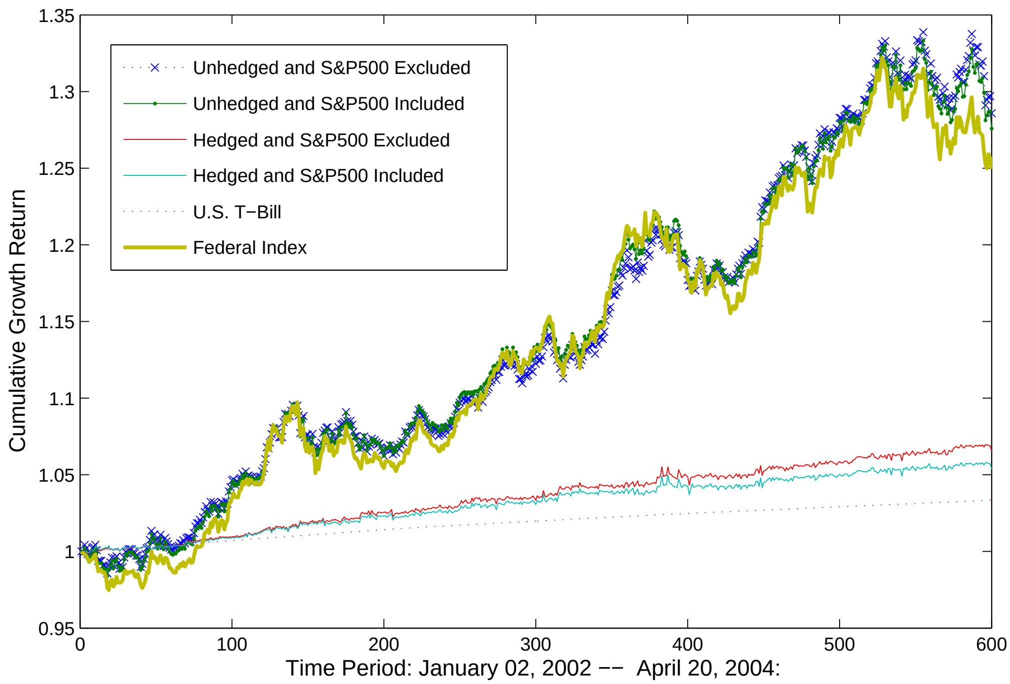

Figure 4. In-sample performance against benchmarks: The effect of including the S&P 500 index returns are not substantial for the two benchmarks which represent the two extremes, either fully hedging or not hedging at all.

Having derived the optimal currency portfolio, we now examine the optimal portfolio of a U.S. investor with a specific performance target. Consider the U.S. rate r_t

and let α be the required excess return over the interest rate of the U.S. dollar. The investor must determine τ_t, the amount to invest in the risky portfolio over time by the formula

$$\tau_t = \frac{\alpha \cdot r_t}{\sum_j p_{i_t j} \cdot \mu_j},$$

where i_t is the regime at time t. Figure 5 presents the performance of a portfolio with $\alpha = 30\%$. The portfolio is close to riskless with a steeper rate of accumulating wealth

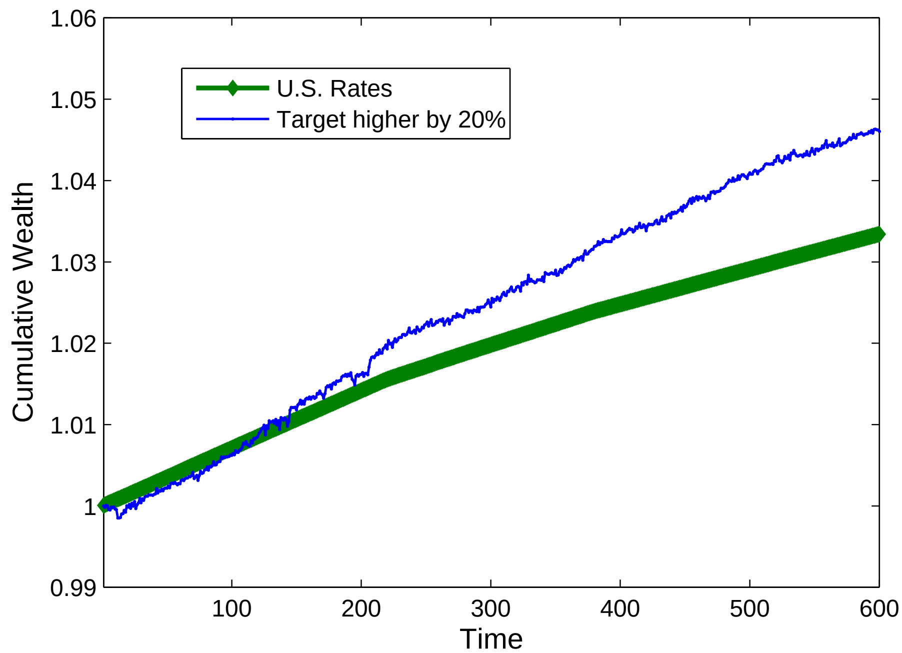

Figure 5. Individual portfolio performance with a target return higher than the risk free rate by $\alpha = 30\%$. With $r_t = 1.5\%$, the optimal weight in the optimal foreign currencies is given as in Figure 6. The average holding of the portfolio in foreign currencies is 0.2997 with small variation between two periods (days), while the minimum is -0.1828 and the maximum 0.5083.

with little holdings in the currency portfolio.

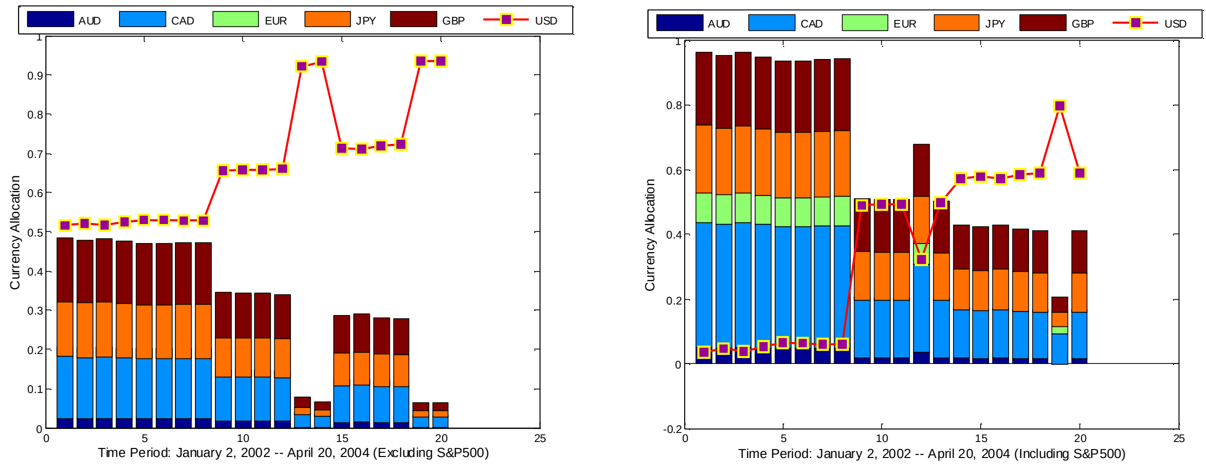

Figure 6. This figure displays the optimal portfolio weights over time for the in-sample data. The weights are plotted every 30 days, so there are 20 data points in total. For the case of including the S&P500 index, it has an expected growth rate of 6.77% with a standard deviation 3.41%. However, for the case of excluding the index, it has an expected return of 1.95% with a standard deviation of 0.56%.

4.5. Out-Of-Sample Performance

To examine out-of-sample performance of the model, we use the remainder of the data set from April 20, 2004 to March 18, 2005 (exactly 238 points). Figure 7 presents the hedged and unhedged optimal portfolio performances for the two estimation approaches. The pattern for the out of sample performance is similar to that for the in-sample data.

Assuming the estimated model is correct, we simulate the most plausible path of regimes over time by using the Viterbi algorithm. For each of the two estimation techniques, the whole data set including the training data is classified into three regimes. The distribution of regimes for the two approaches are (122, 37, 678) and (567, 123,

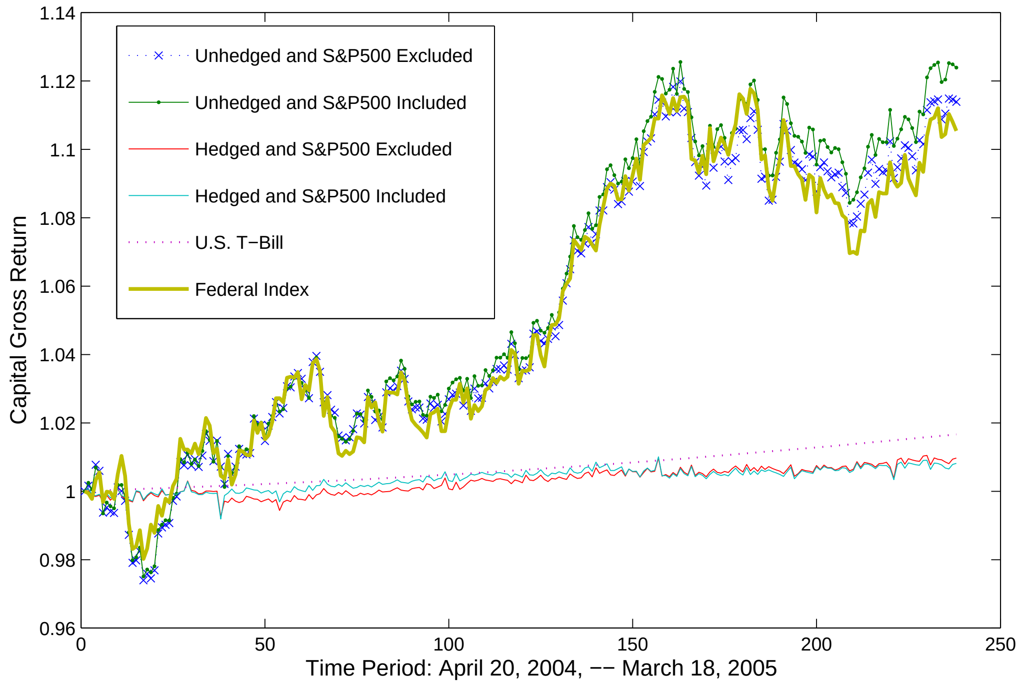

Figure 7. Out-Of-Sample performance. While the hedged portfolio slightly under-performed the U.S. dollar, the unhedged portfolio has a close performance as the currency index. The under-performance of the hedged portfolio is consistent with the recent U.S. interest hikes in 2005.

147), respectively. With a permutation mentioned previously, the percentage of co-incidence of regimes for the two approaches is as high as 72.4% based on a dynamic updating of regimes.

First, we test whether the normality holds for each regime. At the significance level[2], $\alpha = 0.2$, Table VII shows that 12 out of 15 tests fail to reject normality for each of the two estimation approaches.

Tables VIII and IX present the test results for the weak interest rate parity. Only

[2]This is the largest value that the Matlab routine can allow for performing this test. Since our intention is not to reject the null hypothesis, a large value of significance is more convincing when the null hypothesis is not rejected by the test.

a single comparison out of 30 for each of the two approaches rejects equality at the 5% significance level. Furthermore, most of the p-values for the tests are quite high, indicating strong evidence that the weak interest rate parity holds.

For out-of-sample individual portfolio performance, we assume that the investor has a target return of 30% over the risk free rate. Figure 8 presents the performances for both estimation approaches.

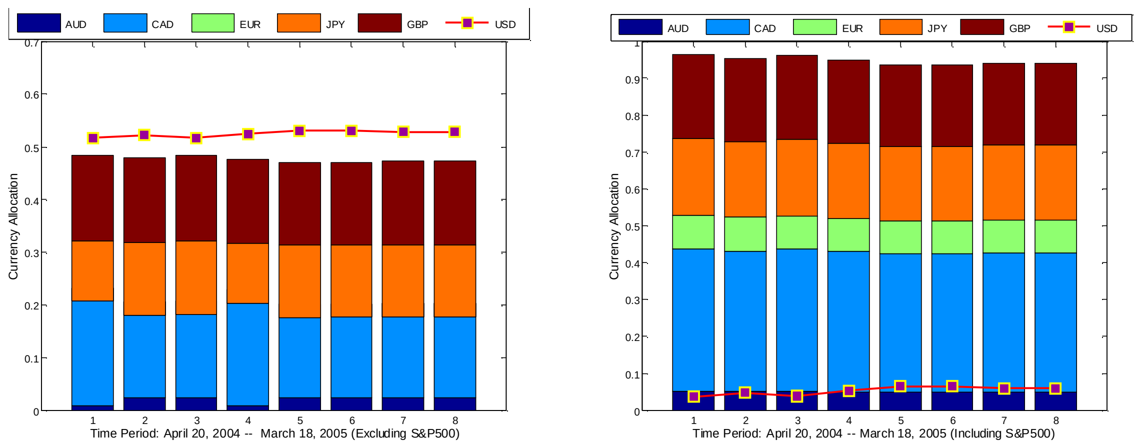

Figure 8. This figure displays the optimal portfolio weights over time for the out-of-sample data. The weights are plotted every 30 days, so there are 8 periods represented in total. For the case of including the S&P500 index, it has an expected growth rate of 10.66% with a standard deviation 5.5%. However, for the case of excluding the index, it has an expected return of 5.5% with a standard deviation of 3.07%.

5. Conclusion

We have developed a three regime switching model for the dynamics of hedged excess returns of currency investments, assuming a weak version of covered interest rate parity. Then, we studied the performance of optimal portfolios in the presence of regimes.

We show that the sample distribution of the hedged excess returns on currency investments are highly skewed and have excess kurtosis. Both Lilliefors and Jarque-Bera tests reject the normality hypothesis of the hedged excess returns for the selected world major currencies. We proposed a three-regime switching model for characterizing the dynamics of the hedged excess returns and a weak version of the covered interest rate parity. The weak version of the parity states that the hedged excess returns are equal in each regime across all currencies. To test the validity of this hypothesis, we performed a series of in-sample and out-of-sample tests. There is strong statistical evidence that the weak interest rate parity holds.

Our results on portfolio performance with switching regimes are promising. Our unhedged optimal currency portfolio is similar to the Federal Exchange Rate Index, and the hedged optimal currency portfolio is similar to the performance of the U.S. treasury bill. These results have important implications for the equilibrium allocation of investments in currency markets.

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A. Estimation for Hidden Markov Models

A.1. Defining a Hidden Markov Model

The Hidden Markov Model is a finite set of states, each of which is associated with a (generally multidimensional) probability distribution. Transitions among the states are governed by a set of probabilities called transition probabilities. In a particular state an outcome or observation can be generated, according to the associated probability distribution. It is only the outcome, not the state, that is visible to an external observer and therefore states are "hidden" to the outside; hence the name Hidden Markov Model.

In order to define an HMM completely, the following elements are needed.

- The number of states of the model, k.
- The number of observation symbols. If the observations are continuous then the set of symbols is infinite.
- A set of state transition probabilities $P = [p_{ij}]$,

  $$p_{ij} = \Pr[M_{t+1} = j \mid M_t = i]$$

  where $M(t)$ denotes the current state. Transition probabilities should satisfy the normal stochastic constraints,

  $$p_{ij} > 0, \qquad \forall 1 \leq i, j \leq 1,$$

  and

  $$\sum_{j=1}^{k} p_{ij} = 1.$$

- A probability distribution in each of the states (in the continuous type, e.g. a normal distribution)

  $$b_j(x_t) = \phi(x_t, \mu_j, \Sigma_j),$$

where μ_j is the mean vector and Σ_j is the variance-covariance matrix for state j. ϕ is a multivariate normal density function and x_t is the observation at time t. Let x be the set of all observations from time 1 to time T.

- The initial state distribution $q = \{q_1, q_2, ...q_k\}$.

The compact form for defining an HMM is

$$\Theta = \{P, B, q\}$$

with P as the transition matrix, B as the conditional distribution parameters, and q the initial state distribution.

A.2. Forward and Backword Algorithms

The forward variable is defined as the probability of the partial observations up to time t, i.e.,

$$\alpha_t(i) = \Pr[M_t = i, x_1, x_2, ..., x_t \mid \Theta].$$

Then, it is easy to see that the following recursion holds true,

$$\alpha_{t+1}(j) = b_j(x_{t+1}) \sum_{i=1}^{k} \alpha_t(i) p_{ij}, \quad 1 \leq j \leq k, 1 \leq t \leq T-1$$

where

$$\alpha_1(j) = q_j b_j(x_1), \quad 1 \leq j \leq k.$$

Using the above recursion, we can calculate

$$\alpha_T(i), \qquad 1 \leq j \leq k$$

and then the required probability is

$$\Pr[x_1, x_2, ..., x_T \mid \Theta] = \sum_{i=1}^{k} \alpha_T(i).$$

Similarly, we can define the backward variable

$$\beta_t(i) = \Pr[x_{t+1}, ..., x_T \mid M_t = i, \Theta].$$

Then,

$$\beta_t(i) = \sum_{j=1}^{k} \beta_{t+1}(j) p_{ij} b_j(x_{t+1}), \quad 1 \leq j \leq k, 1 \leq t \leq T-1,$$

where

$$\beta_T(i) = 1, \quad 1 \leq i \leq k.$$

Hence we obtain

$$\alpha_t(i)\beta_t(i) = \Pr[x, M_t = i \mid \Theta], \quad 1 \leq i \leq k, 1 \leq t \leq T.$$

Therefore

$$\Pr[x \mid \Theta] = \sum_{i=1}^{k} \Pr[x, M_t = i \mid \Theta] = \sum_{i=1}^{k} \alpha_t(i)\beta_t(i).$$

A.3. Baum-Welch Algorithm

Using calculus we maximize the auxiliary quantity

$$\max_{\bar{\Theta}} \quad Q(\Theta, \bar{\Theta}) = \sum_{y} p(y \mid x, \Theta) \ln[p(x, y, \bar{\Theta})]$$

We need to calculate the posterior probability of a prevailing state, $q_t(i) = \Pr[M_t = i \mid x, \Theta]$. Let,

$$\xi_t(i, j) = \Pr[M_t = i, M_{t+1} = j \mid x, \Theta].$$

This is the same as

$$\xi_t(i,j) = \frac{\Pr[M_t = i, M_{t+1} = j, x|\Theta]}{\Pr[x|\ \Theta]}.$$

Using forward and backward variables, this can be expressed by

$$q_t(i) = \frac{\alpha_t(i)\beta_t(i)}{\sum_{j=1}^{k} \alpha_t(j)\beta_t(j)}.$$

One can see that the relationship between $q_t(i)$ and $\xi_t(i,j)$ is

$$q_t(i) = \sum_{j=1}^{k} \xi_t(i,j), \qquad 1 \leq j \leq k, 1 \leq t \leq T.$$

A.4. Parameter Updating

$$\hat{p}_{ij} = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} q_t(i)}, \qquad 1 \leq i \leq k, 1 \leq j \leq k.$$

$$\hat{q}_i = q_1(i), \quad 1 \leq i \leq k.$$

A.5. Maximum Likelihood

The maximization step uses the updated parameters, Θ, as known, and maximizes the log likelihood, $Q(\Theta, \bar{\Theta})$, over $\bar{\Theta}$. Continuing the iteration procedure guarantees a local convergence.

Table I: Sample Statistics and Normality Tests

The daily returns and the standard deviations are measured using a "base point". The skewness and excess kurtosis for the excess returns on the five currencies and the S&P500 index are statistically not equal to zero, indicating that the returns on currency investments are not evidently normally distributed. Both the Lilliefors and Jarque Bera tests reject the null hypothesis that the hedged risk premium, R_{tj}, are normally distributed. There are some evidence showing that the currencies are likely to be negatively correlated to the equity market. The EURO, JPY, and GBP are generally moving together against the equity market, while the AUD and CAD are moving with the U.S. market, but not in a significant scale. This implies a significant opportunity for diversification.

Statistics	**S&P500**	**AUD**	**CAD**	**EURO**	**JPY**	**GBP**
Mean Return	-0.45	1.28	0.6769	0.05	-0.48	-2.02
Standard Deviation	129.74	32.39	20.53	25.52	23.98	49.09
Correlation	1.0000	0.1500	0.0948	-0.0405	-0.0932	-0.0512
	0.1500	1.0000	0.4297	0.3347	0.0259	0.3377
	0.0948	0.4297	1.0000	0.2136	0.0996	0.2103
	-0.0405	0.3347	0.2136	1.0000	0.2965	0.6038
	-0.0932	0.0259	0.0996	0.2965	1.0000	0.3223
	-0.0512	0.3377	0.2103	0.6038	0.3223	1.0000
Skewness	0.27	0.28	-0.11	-0.07	0.09	0.13
Excess Kurtosis	1.55	7.58	2.96	4.81	6.82	0.61
Lilliefors Test –	1	1	1	1	1	1
test statistic	0.0593	0.0744	0.0538	0.0829	0.0881	0.0824
Critcal Value	0.0306	0.0306	0.0306	0.0306	0.0306	0.0306
Jarque Bera Test	1	1	1	1	1	1

Table II: Estimation for Means and Variance-Covariances (I)

This table presents the estimation results of the initial probability q, the mean excess returns of the five currency returns and their variance-covariance matrices, and the transition matrix P. The S&P500 index returns are not used in this estimation procedure.

Statistics	AUD	CAD	EURO	JPY	GBP
Regime 1		$\mu_1 = 3.2216,$		$q_1 = 0.$	
Standard Deviations	38.3662	27.5456	36.2456	40.3105	33.2757
Correlations	1.0000	0.2599	0.6039	0.1749	0.7083
	0.2599	1.0000	0.4050	0.1699	0.2704
	0.6039	0.4050	1.0000	0.4054	0.7297
	0.1749	0.1699	0.4054	1.0000	0.0798
	0.7083	0.2704	0.7297	0.0798	1.0000
Regime 2		$\mu_2 = -1.8641,$		$q_2 = 1.$	
Standard Deviations	77.7775	42.8397	56.5907	47.7489	38.5394
Correlations	1.0000	0.6492	0.0820	-0.2665	0.0894
	0.6492	1.0000	-0.1055	-0.0583	0.1168
	0.0820	-0.1055	1.0000	-0.0330	0.2967
	-0.2665	-0.0583	-0.0330	1.0000	0.6686
	0.0894	0.1168	0.2967	0.6686	1.0000
Regime 3		$\mu_3 = 0.2086,$		$q_3 = 0.$	
Standard Deviations	22.7248	14.9195	16.7986	14.0490	14.1620
Correlations	1.0000	0.3611	0.3537	0.1428	0.2077
	0.3611	1.0000	0.3008	0.1592	0.2055
	0.3537	0.3008	1.0000	0.4825	0.7050
	0.1428	0.1592	0.4825	1.0000	0.4232
	0.2077	0.2055	0.7050	0.4232	1.0000
Transition Probabilities		0.6141	0.0610	0.3249	
		0.0000	0.6556	0.3444	
		0.0890	0.0156	0.8954	

Table III: Estimation for Means and Variance-Covariances (II)

This table presents the estimation results of the same parameters as in Table II but with the S&P500 index return as an exogenous factor. Means and standard deviations are represented by based points.

Statistics	AUD	CAD	EURO	JPY	GBP
Regime 1		$\mu_1 = 0.2480,$		$q_1 = 0.$	
Standard Deviations	24.7493	16.9046	19.7100	15.4814	15.9288
Correlations	1.0000	0.3720	0.4629	0.2282	0.3235
	0.3720	1.0000	0.3971	0.3083	0.2809
	0.4629	0.3971	1.0000	0.4899	0.6819
	0.2282	0.3083	0.4899	1.0000	0.3710
	0.3235	0.2809	0.6819	0.3710	1.0000
Regime 2		$\mu_2 = 0.8247,$		$q_2 = 0.$	
Standard Deviations	58.1820	35.2593	46.8211	46.1690	38.6658
Correlations	1.0000	0.4936	0.2985	-0.0576	0.3863
	0.4936	1.0000	0.1311	0.0471	0.2002
	0.2985	0.1311	1.0000	0.1998	0.5597
	-0.0576	0.0471	0.1998	1.0000	0.2936
	0.3863	0.2002	0.5597	0.2936	1.0000
Regime 3		$\mu_3 = 0.1826,$		$q_3 = 1.$	
Standard Deviations	17.3314	10.7179	9.8515	13.9717	9.8244
Correlations	1.0000	0.2025	-0.1253	-0.0624	-0.0850
	0.2025	1.0000	-0.2009	-0.3405	-0.1341
	-0.1253	-0.2009	1.0000	0.4876	0.6102
	-0.0624	-0.3405	0.4876	1.0000	0.4000
	-0.0850	-0.1341	0.6102	0.4000	1.0000
Transition Probabilities		0.8475	0.0977	0.0548	
		0.3024	0.6308	0.0668	
		0.1462	0.0483	0.8056	

Table IV: t-Tests for the Weak Interest Rate Parity (In-Sample Data with S&P 500 Index Excluded for Parameter Estimation))

Regimes	Currencies (i, j)	Test Result	p-Value	t-Statistics	Confidence Interval (5%)
Regime 1	(2,1)	0	0.1492	-1.4486	(-0.0019, 0.0005)
	(3,1)	0	0.0932	-1.6876	(-0.0023, 0.0004)
	(4,1)	1.0000	0.0230	-2.2923	(-0.0028, 0.0000)
	(5,1)	0	0.5127	-0.6559	(-0.0016, 0.0009)
	(3,2)	0	0.6450	-0.4616	(-0.0014, 0.0010)
	(4,2)	0	0.2357	-1.1897	(-0.0019, 0.0006)
	(5,2)	0	0.4249	0.7998	(-0.0007, 0.0015)
	(4,3)	0	0.5031	-0.6709	(-0.0018, 0.0010)
	(5,3)	0	0.2617	1.1258	(-0.0007, 0.0019)
	(5,4)	0	0.0764	1.7820	(-0.0003, 0.0024)
Regime 2	(2,1)	0	0.8796	0.1520	(-0.0036, 0.0041)
	(3,1)	0	0.8074	0.2447	(-0.0038, 0.0046)
	(4,1)	0	0.8604	0.1766	(-0.0037, 0.0043)
	(5,1)	0	0.7635	0.3021	(-0.0033, 0.0043)
	(3,2)	0	0.8865	0.1433	(-0.0029, 0.0032)
	(4,2)	0	0.9665	0.0421	(-0.0027, 0.0028)
	(5,2)	0	0.8226	0.2250	(-0.0022, 0.0027)
	(4,3)	0	0.9208	-0.0999	(-0.0033, 0.0031)
	(5,3)	0	0.9673	0.0412	(-0.0029, 0.0030)
	(5,4)	0	0.8692	0.1654	(-0.0025, 0.0029)
Regime 3	(2,1)	0	0.6982	0.3879	(-0.0002, 0.0003)
	(3,1)	0	0.9855	0.0182	(-0.0003, 0.0003)
	(4,1)	0	0.8578	0.1793	(-0.0003, 0.0003)
	(5,1)	0	0.9316	0.0858	(-0.0003, 0.0003)
	(3,2)	0	0.6529	-0.4499	(-0.0003, 0.0002)
	(4,2)	0	0.7788	-0.2810	(-0.0002, 0.0002)
	(5,2)	0	0.6884	-0.4012	(-0.0003, 0.0002)
	(4,3)	0	0.8440	0.1968	(-0.0002, 0.0003)
	(5,3)	0	0.9348	0.0819	(-0.0002, 0.0002)
	(5,4)	0	0.9006	-0.1250	(-0.0002, 0.0002)

Table V: t-Tests for the Weak Interest Rate Parity (In-Sample Data with S&P 500 Index Included for Parameter Estimation)

Regimes	Currencies (i, j)	Test Result	p-Value	t-Statistics	Confidence Interval (5%)
Regime 1	(2,1)	0	0.6878	0.4021	(-0.0003, 0.0004)
	(3,1)	0	0.8200	0.2276	(-0.0003, 0.0004)
	(4,1)	0	0.8440	0.1969	(-0.0003, 0.0004)
	(5,1)	0	0.9170	0.1043	(-0.0003, 0.0004)
	(3,2)	0	0.8531	-0.1852	(-0.0003, 0.0003)
	(4,2)	0	0.7866	-0.2708	(-0.0003, 0.0002)
	(5,2)	0	0.6985	-0.3875	(-0.0003, 0.0002)
	(4,3)	0	0.9549	-0.0566	(-0.0003, 0.0003)
	(5,3)	0	0.8703	-0.1633	(-0.0003, 0.0003)
	(5,4)	0	0.9036	-0.1211	(-0.0003, 0.0003)
Regime 2	(2,1)	0	0.2664	-1.1144	(-0.0024, 0.0009)
	(3,1)	0	0.1775	-1.3532	(-0.0028, 0.0008)
	(4,1)	0	0.0666	-1.8444	(-0.0032, 0.0004)
	(5,1)	0	0.6771	-0.4170	(-0.0020, 0.0014)
	(3,2)	0	0.6696	-0.4273	(-0.0017, 0.0012)
	(4,2)	0	0.2973	-1.0449	(-0.0020, 0.0008)
	(5,2)	0	0.3743	0.8905	(-0.0008, 0.0017)
	(4,3)	0	0.5883	-0.5422	(-0.0020, 0.0012)
	(5,3)	0	0.2383	1.1827	(-0.0007, 0.0022)
	(5,4)	0	0.0752	1.7885	(-0.0003, 0.0026)
Regime 3	(2,1)	0	0.4088	0.8273	(-0.0003, 0.0006)
	(3,1)	0	0.4511	0.7547	(-0.0003, 0.0005)
	(4,1)	0	0.3307	0.9746	(-0.0003, 0.0006)
	(5,1)	0	0.5764	0.5593	(-0.0003, 0.0005)
	(3,2)	0	0.8961	-0.1307	(-0.0003, 0.0003)
	(4,2)	0	0.7913	0.2649	(-0.0003, 0.0004)
	(5,2)	0	0.6904	-0.3987	(-0.0003, 0.0002)
	(4,3)	0	0.6981	0.3884	(-0.0003, 0.0004)
	(5,3)	0	0.7772	-0.2832	(-0.0003, 0.0002)
	(5,4)	0	0.5353	-0.6207	(-0.0004, 0.0002

Table VI: Optimal Portfolio Weights

The numbers in brackets are the portfolio weights when the S&P 500 index returns are included for estimating parameters.

Portfolio	AUD	CAD	EURO	JPY	GBP
Regime (1)	0.0178	0.4596	-0.0503	0.2386	0.3342
	(0.0371	0.3645	-0.0138	0.2919	0.3203)
Regime (2)	-0.0045	0.4148	0.2350	0.2931	0.0616
	(-0.0081	0.4520	0.1100	0.2250	0.2211)
Regime (3)	0.0493	0.3799	-0.0534	0.2900	0.3342
	(0.0520	0.4008	0.0950	0.2159	0.2362)
Steady State	0.0243	0.4032	0.0520	0.2479	0.2727
	(0.0214	0.4041	0.0542	0.2528	0.2675)
Naive	0.0210	0.4045	0.0550	0.2534	0.2662

Table VII: Lilliefors and Jarque-Bera Tests for Normality ($\alpha = 0.2$)

With a significance level 0.05, all testing results are in favor of the null hypothesis that all returns on the currencies are normally distributed within each regime. The skewness and the excess kurtosis are close to zero. The mean absolute values of the skewness and the excess kurtosis for all regimes and all currencies are 0.2240 and 0.6382, respectively.

IN := In sample, OUT :=Out of Sample.

YES :=Including the S&P500, NO := Excluding the S&P500

Regimes	Currencies	(NO, IN)	(YES, IN)	(NO, OUT)	(YES, OUT)
Regime 1	AUD	0	1	0	1
	CAD	0	0	1	0
	EUR	1	0	1	0
	JPY	0	1	0	0
	GBP	0	0	0	1
Regime 2	AUD	0	0	0	0
	CAD	1	0	1	0
	EUR	0	0	0	0
	JPY	0	1	0	1
	GBP	0	0	0	0
Regime 3	AUD	0	0	0	0
	CAD	1	0	0	0
	EUR	0	0	0	0
	JPY	0	0	0	0
	GBP	1	0	0	0

Table VIII: t-Tests for the Weak Interest Rate Parity (Out-Of-Sample with S&P500 Index Excluded for Parameter Estimation)

Regimes	Currencies (i, j)	Test Result	p-Value	t-Statistics	Confidence Interval (5%)
Regime 1	(2,1)	0	0.1522	-1.4363	(-0.0015, 0.0002)
	(3,1)	0	0.0718	-1.8080	(-0.0019, 0.0001)
	(4,1)	1.0000	0.0138	-2.4800	(-0.0024, -0.0003)
	(5,1)	0	0.4893	-0.6924	(-0.0013, 0.0006)
	(3,2)	0	0.5630	-0.5792	(-0.0011, 0.0006)
	(4,2)	0	0.1503	-1.4431	(-0.0016, 0.0002)
	(5,2)	0	0.4355	0.7812	(-0.0005, 0.0011)
	(4,3)	0	0.3966	-0.8492	(-0.0014, 0.0006)
	(5,3)	0	0.2155	1.2418	(-0.0003, 0.0015)
	(5,4)	1.0000	0.0462	2.0036	(0.0000, 0.0020)
Regime 2	(2,1)	0	0.9490	0.0641	(-0.0030, 0.0031)
	(3,1)	0	0.8992	0.1271	(-0.0031, 0.0035)
	(4,1)	0	0.9463	-0.0676	(-0.0033, 0.0031)
	(5,1)	0	0.7336	0.3417	(-0.0027, 0.0038)
	(3,2)	0	0.9254	0.0940	(-0.0023, 0.0025)
	(4,2)	0	0.8534	-0.1855	(-0.0024, 0.0020)
	(5,2)	0	0.6898	0.4007	(-0.0018, 0.0027)
	(4,3)	0	0.8058	-0.2467	(-0.0029, 0.0023)
	(5,3)	0	0.7981	0.2567	(-0.0023, 0.0030)
	(5,4)	0	0.5901	0.5412	(-0.0018, 0.0031)
Regime 3	(2,1)	0	0.6718	0.4238	(-0.0002, 0.0003)
	(3,1)	0	0.9985	0.0018	(-0.0002, 0.0002)
	(4,1)	0	0.7712	0.2908	(-0.0002, 0.0002)
	(5,1)	0	0.8040	0.2482	(-0.0002, 0.0002)
	(3,2)	0	0.5973	-0.5284	(-0.0002, 0.0001)
	(4,2)	0	0.8555	-0.1821	(-0.0002, 0.0001)
	(5,2)	0	0.8099	-0.2405	(-0.0002, 0.0001)
	(4,3)	0	0.7161	0.3638	(-0.0001, 0.0002)
	(5,3)	0	0.7562	0.3105	(-0.0001, 0.0002)
	(5,4)	0	0.9532	-0.0587	(-0.0002, 0.0001)

Table IX: t-Tests for the Weak Interest Rate Parity (Out-Of-Sample with S&P500 Index Included for Parameter Estimation)

Regimes	Currencies (i, j)	Test Result	p-Value	t-Statistics	Confidence Interval (5%)
Regime 1	(2,1)	0	0.6379	0.4708	(-0.0002, 0.0003)
	(3,1)	0	0.8578	0.1793	(-0.0002, 0.0003)
	(4,1)	0	0.6934	0.3944	(-0.0002, 0.0003)
	(5,1)	0	0.7893	0.2672	(-0.0002, 0.0003)
	(3,2)	0	0.7313	-0.3434	(-0.0002, 0.0002)
	(4,2)	0	0.9191	-0.1016	(-0.0002, 0.0002)
	(5,2)	0	0.7829	-0.2756	(-0.0002, 0.0002)
	(4,3)	0	0.8023	0.2505	(-0.0002, 0.0002)
	(5,3)	0	0.9248	0.0945	(-0.0002, 0.0002)
	(5,4)	0	0.8626	-0.1731	(-0.0002, 0.0002)
Regime 2	(2,1)	0	0.2041	-1.2733	(-0.0020, 0.0004)
	(3,1)	0	0.1196	-1.5621	(-0.0024, 0.0003)
	(4,1)	1.0000	0.0200	-2.3417	(-0.0029, -0.0003)
	(5,1)	0	0.6929	-0.3954	(-0.0015, 0.0010)
	(3,2)	0	0.6177	-0.4997	(-0.0013, 0.0008)
	(4,2)	0	0.1359	-1.4961	(-0.0019, 0.0003)
	(5,2)	0	0.3000	1.0387	(-0.0005, 0.0015)
	(4,3)	0	0.3660	-0.9057	(-0.0017, 0.0006)
	(5,3)	0	0.1687	1.3805	(-0.0003, 0.0019)
	(5,4)	1.0000	0.0230	2.2876	(0.0002, 0.0025)
Regime 3	(2,1)	0	0.4526	0.7521	(-0.0002, 0.0005)
	(3,1)	0	0.5243	0.6375	(-0.0002, 0.0004)
	(4,1)	0	0.3686	0.9004	(-0.0002, 0.0005)
	(5,1)	0	0.6498	0.4545	(-0.0002, 0.0004)
	(3,2)	0	0.8491	-0.1904	(-0.0003, 0.0002)
	(4,2)	0	0.8015	0.2517	(-0.0002, 0.0003)
	(5,2)	0	0.6578	-0.4434	(-0.0003, 0.0002)
	(4,3)	0	0.6668	0.4309	(-0.0002, 0.0003)
	(5,3)	0	0.7875	-0.2698	(-0.0002, 0.0002)
	(5,4)	0	0.5138	-0.6537	(-0.0004, 0.0002)

Table X: F-Tests for the Weak Interest Rate Parity (with S&P500 Index excluded for Parameter Estimation)

The p-values for the F-test in the three regimes imply that none of the null hypotheses are rejected at the level of 5%. Actually the p-values in two of the three regimes are as high as 0.99, which shows a strong support for the weak interest rate parity hypothesis.

Regime 1:					
Source	SS	df	MS	F	Prob>F
Columns	0.00010454	4	2.6134e-005	1.8932	0.11053
Error	0.006281	455	1.3804e-005		
Total	0.0063855	459			
Regime 2:					
Source	SS	df	MS	F	Prob>F
Columns	4.8361e-006	4	1.209e-006	0.036006	0.9975
Error	0.0055405	165	3.3579e-005		
Total	0.0055454	169			
Regime 3:					
Source	SS	df	MS	F	Prob>F
Columns	7.5534e-007	4	1.8883e-007	0.065311	0.99217
Error	0.0068379	2365	2.8913e-006		
Total	0.0068387	2369			

Table XI: F-Tests for the Weak Interest Rate Parity (with S&P500 Index included for Parameter Estimation)

The p-values for the F-test in the three regimes imply that none of the null hypotheses are rejected at the level of 5%. Actually the p-values in all of the three regimes are greater than 0.2 with 0.9975 in one regime, which shows a strong support for the weak interest rate parity hypothesis.

Regime 1:					
Source	SS	df	MS	F	Prob>F
Columns	8.138e-007	4	2.0345e-007	0.057071	0.99396
Error	0.0064881	1820	3.5649e-006		
Total	0.0064889	1824			
Regime 2:					
Source	SS	df	MS	F	Prob>F
Columns	0.00012787	4	3.1967e-005	1.4354	0.22103
Error	0.011135	500	2.2271e-005		
Total	0.011263	504			
Regime 3:					
Source	SS	df	MS	F	Prob>F
Columns	2.6463e-006	4	6.6157e-007	0.41586	0.79728
Error	0.0010579	665	1.5909e-006		
Total	0.0010606	669			