Complex Times: Asset Pricing and Conditional Moments under Non-Affine Diffusions*

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Abstract

We develop methods for the approximation of solutions to the Chapman-Kolmogorov backward and Feynman-Kac partial differential equations, where the method of approximation is accurate for very long time horizons. When an underlying economy is modeled by a diffusion process, asset prices and conditional expectations of the state variables can be found as solutions to these partial differential equations. However, for all but a few simple cases, solutions cannot be found explicitly in closed form. The form of these equations suggests constructing a power series in the time variable as a method of solution. However, the convergence properties of such power series solutions are often quite poor. We examine the problem of determining the convergence properties of power series solutions, and introduce a parameterized family of non-affine transformations of the time variable that can substantially improve the rate of convergence for long time horizons. In some cases, the approximations converge uniformly (in time) to the true (but unknown) solutions for arbitrarily large time horizons. The ability to approximate solutions accurately and in closed form simplifies the estimation of non-affine continuous-time term structure models, since the bond pricing problem must be solved for many different parameter vectors during a typical estimation procedure.

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1 Introduction

Many applications in financial economics require solutions to second order parabolic differential equations with a boundary condition. Continuous-time models are often expressed as solutions to stochastic differential equations. Estimation of the parameters of such a model can be performed by a variety of techniques, including maximum likelihood or method of moments. Likelihood functions solve the Chapman-Kolmogorov forward and backward equations; conditional moments solve the backward equation. Prices of derivative securities with European-style exercise are solutions to the Feynman-Kac equation with a final condition (which can effectively be turned into an initial condition by changing the time variable). In term structure models, bonds are often considered derivatives on the interest rate, and are therefore solutions to the Feynman-Kac equation. In some estimation problems, both equations are encountered. For example, a model may be written in terms of a set of latent variables not directly observed; rather, the observed quantities are prices of derivatives based on those variables (e.g., option or bond prices). In this case, solutions to the Feynman-Kac equation are observed; the values of the latent state variables can be inferred from the prices of the observed instruments. The likelihood of successive observations of the state vector can then be calculated for maximum likelihood estimation; the sample values of conditional moments can be calculated for method of moment based estimation techniques. Estimation typically proceeds by repeating this operation for many values of the model parameters until the likelihood is maximized or the conditional moments are matched.

However, the class of continuous-time models for which conditional moments, likelihood functions, or derivative prices are known in closed form is quite limited. Likelihoods, conditional moments, and prices of standard derivatives can all be found in closed form for the geometric Brownian motion model of equity prices used by Black and Scholes (1973). In the term structure models of Vasicek (1977) and Cox, Ingersoll, and Ross (1985), likelihoods, conditional moments, and bond prices are all known in closed form.[1] However, more complicated models almost always lose some of the tractability of these early models. For example, in the stochastic volatility model of Heston (1993), option prices must be found through numeric methods, such as Fourier transforms. In the general affine yield models of Duffie and Kan (1996), conditional moments of the state variables are known in closed form, but (except for a few restricted special cases) neither bond prices nor likelihoods can be found explicitly. Estimation of such models therefore typically requires some form of approximation or numerical analysis. The affine structure of this class of models ensures that there are rapid numeric techniques for the calculation of bond prices, which are practical even if bond prices must be calculated many times during estimation. Estimation for this class of models has been by simulated method of moments, from Dai and Singleton (2000), by quasi-maximum likelihood, from Duffee (2002), and by closed-form approximation to likelihoods, as in Aït-Sahalia and Kimmel (2005), Cheridito, Filipović, and Kimmel (2005), Thompson (2004), and Mosburger and Schneider (2005).

Non-linear models are much less common in the literature. For some classes of non-linear models, such as Ahn and Gao (1999), Constantinides (1992), and Ahn, Dittmar, and Gallant (2002), bond prices are known

[1] We use the term "closed form" to include expressions such as the cumulative Gaussian distribution function and modified Bessel functions of the first kind. With a stricter definition of "closed form," the class of models for which closed form solutions exist is even more limited.

in closed-form. In general, though, numeric analysis of non-linear models precludes practical use. Some techniques, such as Aït-Sahalia (2002), can be applied to non-linear models if the state vector is directly observed. This technique develops a series of approximations that converge to the true (but unknown) likelihood function. However, this approach does not solve the problem of extracting the values of the state variables from observed prices. It is possible to construct series approximations to solutions to the Feynman-Kac differential equation, in a manner similar to the method used to construct likelihood expansions. However, it is necessary for the likelihood expansions to converge rapidly only for relatively small values of the time horizon; if data are observed at a weekly frequency, for example, the accuracy of the likelihood approximations at a frequency of one year (if the series converges at all for this time horizon) is irrelevant. By contrast, any approximation to solutions of the Feynman-Kac differential equation must be accurate at much longer time horizons to be of practical use; if bonds with maturities of 10 years are used, the approximations must be accurate at this horizon. Series expansions to likelihoods have proven useful at the short horizons at which data are typically observed (see, for example, Aït-Sahalia (1999), Aït-Sahalia and Kimmel (2005), or Cheridito, Filipović, and Kimmel (2005)), and Itô-Taylor expansions have long been used (see, for example, Kloeden and Platen (1999)). However, this method of expansion does not extend to general solutions to the Feynman-Kac equation in a straightforward manner, and power series approximations to such solutions often do not converge for large time horizons. Even if they do, such convergence is often so slow as to make the technique impractical. Thus, despite recent advances in estimation of non-linear models, estimation in which the values of latent state variables must be extracted from observed prices remains stymied.

We therefore develop a technique for the series approximations of solutions to the Chapman-Kolmogorov backward and Feynman-Kac equations that, in many cases, converge for very long time horizons, and that sometimes also converge uniformly for arbitrarily large time horizons. Our approach is similar to the standard approach of calculating power series expansions through a recursive relation. However, we first transform the time variable to improve the convergence of the series of approximations. Applying this approach to a number of diffusion processes, we are able to construct a large class of final conditions such that the corresponding moments are analytic in the time variable, and also a large class of non-affine term structure models in which bond prices are analytic in maturity. Furthermore, the use of the non-affine time transformation allows approximation to these solutions for very long time horizons.

The rest of this paper is organized as follows. In Section 2, we discuss the general problem of constructing series solutions to the Feynman-Kac equation (with the Chapman-Kolmogorov backward equation as a special case), and illustrate the problems with this approach. In Section 3, we consider a parameterized family of transformations of the time variable and develop power series solutions to the Feynman-Kac equation with improved convergence properties. In Section 4, we characterize a particular family of processes and interest rate specifications for which our technique can be applied, and determine the convergence properties of power series expansions to conditional moments and bond prices. In Section 5, we consider a number of examples to demonstrate the applicability and to test the accuracy of the technique. Finally, Section 6 concludes.

2 Series Solutions

We consider an N-dimensional diffusion process:

$$dX_t = \mu(X_t) + \sigma(X_t)\, dW_t \tag{2.1}$$

with initial condition $X_t = X_0$, where W_t is an N-dimensional Brownian motion. We assume that $\mu(X_t)$ and $\sigma(X_t)$ are chosen so that a unique solution X_t exists. There are numerous criteria for existence and uniqueness of solutions to stochastic differential equations in the literature; see, for example, Karatzas and Shreve (1991), Stroock and Varadhan (1979), or Liptser and Shiryaev (2001). For now, we do not specify the particular existence and uniqueness requirements imposed, so as not to result in a loss of generality. We are interested in finding expectations of the following form, conditional on knowledge of the state vector at an earlier time:

$$f(\Delta, x) = E\left[e^{-\int_t^{t+\Delta} r(X_u)du} g(X_{t+\Delta}) \mid X_t = x\right] \tag{2.2}$$

for some scalar-valued functions $r(x)$ and $g(x)$, and for some time horizon $\Delta \geq 0$. Under relatively mild technical assumptions, the solution $f(\Delta, X_t)$ to the probabilistic problem is also a solution to the partial differential equation:

$$\frac{\partial f}{\partial \Delta}(\Delta, x) = \sum_{i=1}^{N} [\mu(x)]_i \frac{\partial f}{\partial x_i}(\Delta, x) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \left[\sigma^2(x)\right]_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(\Delta, x) - r(x) f(\Delta, x) \tag{2.3}$$

with the final condition:

$$f(0, x) = g(x) \tag{2.4}$$

Subject only to technical regularity conditions, a solution of Equation 2.3 is the price of a derivative instrument with final payoff $g(x)$. The Chapman-Kolmogorov backward equation is obtained by setting $r(x) = 0$; in this case, solutions to the partial differential equation are conditional expectations (also subject to technical regularity conditions).

The form of this partial differential equation suggests that the solution can be written as a power series in Δ, centered at zero:

$$f(\Delta, x) = \sum_{i=0}^{\infty} a_i(x) \frac{\Delta^i}{i!} \tag{2.5}$$

The final condition requires:

$$a_0(x) = g(x) \tag{2.6}$$

Substitution of the proposed solution into Equation 2.3 shows that the functions $a_n(x)$ for $n \geq 1$ must satisfy a recursive relation:

$$a_n(x) = \sum_{i=1}^{N} [\mu(x)]_i \frac{\partial a_{n-1}}{\partial x_i}(x) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\sigma^2(x)\right]_{ij} \frac{\partial^2 a_{n-1}}{\partial x_i \partial x_j}(x) - r(x)\, a_{n-1}(x) \tag{2.7}$$

Provided $g(x)$, $\mu(x)$, $\sigma^2(x)$, and $r(x)$ are all infinitely differentiable in a neighborhood of x,[2] the coefficients as defined above all exist. A related approach[3] has been used to approximate likelihood functions for scalar diffusions (including the non-affine case); see Aït-Sahalia (2002). The approach can be applied to a wide class of diffusion processes. This type of expansion can be interpreted as the deterministic part of the stochastic Itô-Taylor expansions described in, for example, Kloeden and Platen (1999).

Any power series trivially converges at the point of expansion. However, for large values of Δ (and possibly for any non-zero values of Δ), the proposed power series solutions may converge very slowly, or not at all. (Worse still, it may converge to the wrong function.) For the probabilistic problem, we are typically interested in solutions of the differential equation for non-negative time horizons, i.e., $\Delta \in [0, +\infty)$. Even when the coefficients of the differential equation and the final condition satisfy strong smoothness conditions, a power series solution may fail to converge for some non-negative time horizons. To illustrate some of the problems that can occur, we consider the very simple special case of finding conditional moments of a function of a Brownian motion. In this case, the coefficients of Equation 2.3 are given by:

$$\mu(x) = 0 \tag{2.8}$$

$$\sigma(x) = 1 \tag{2.9}$$

$$r(x) = 0 \tag{2.10}$$

We seek the conditional moment:

$$f(\Delta, x) = E\left[e^{cX_{t+\Delta}^2} \mid X_t = x\right] \tag{2.11}$$

for some $c > 0$. The solution is known in closed form:

$$f(\Delta, x) = \frac{e^{\frac{cx^2}{1-2\Delta c}}}{\sqrt{1-2\Delta c}} \tag{2.12}$$

[2] These conditions are sufficient, but not necessary, for the existence of the coefficients $a_n(x)$. For example, if $g(x)$ and $\mu(x)$ are both affine in x, then the coefficients $a_n(x)$ can be found even if $\sigma^2(x)$ is not differentiable. The coefficients $a_n(x)$ can even be found in some cases in which the coefficients of the partial differential equation do not even specify a valid diffusion process. Of course, existence of the coefficients does not guarantee convergence of the series anywhere but the origin. In general, series solutions to the prices of at-the-money standard call and put options cannot be found; series solutions to the prices of other call and put options actually converge to a function that is not equal to the option price.

[3] Although the approach may seem similar at first glance, there are important differences. Aït-Sahalia (2002) finds series approximations to likelihood functions rather than conditional moments or asset prices, and the series approximations include some leading terms that do not conform to the power series form. The likelihood function of a diffusion can never be expressed as a pure power series in Δ about the origin, since there is always a singularity at $\Delta = 0$. Furthermore, as we see in Sections 4 and 5, the analyticity of a solution depends critically on the final condition, an aspect that is wholly absent from the problem of finding a likeklihood function.

This function has a singularity at $\Delta = 1/(2c)$. Consequently, a power series expansion about $\Delta = 0$ converges only on the interval $\left(-\frac{1}{2c}, +\frac{1}{2c}\right)$ (and possibly also at $-1/(2c)$, the lower endpoint of the interval). Arguably, this behavior is appropriate for a power series expansion in this case, since the conditional expectation approaches positive infinity as Δ approaches $1/(2c)$ from below; the tails of the final condition go to infinity too rapidly for the existence of a conditional moment for $\Delta \geq 1/(2c)$.

We can also consider the case $c < 0$. The function $f(\Delta, x)$ specified by Equation 2.12 is the conditional moment for this case as well, and has a singularity at $\Delta = 1/(2c) < 0$. The solution is well-behaved for all $\Delta \geq 0$, which corresponds to the time horizons that are typically of interest in financial and economic applications. However, the interval of convergence of a power series expansion about the origin is $\left(+\frac{1}{2c}, -\frac{1}{2c}\right)$ (and possibly also at $-1/(2c)$, the upper endpoint of the interval). Even if we are interested only in the case $\Delta \geq 0$, the existence of the singularity in the extension of $f(\Delta, x)$ to negative values of Δ (which are not of interest) causes non-convergence of a power series approximation for positive values of Δ (which are of interest). So even well-behaved conditional moments may fail to have convergent power series expansions, because of the behavior of the extension of the conditional moment function to negative time horizons. In this case, the power series fails to converge because the tails of the initial condition go too rapidly not to infinity, but to zero.

The analogy to heat diffusion provides some intuition for the previous case. Is it possible to specify a distribution of heat through a uniform and infinitely long rod at some specified time $\Delta < 0$, such that the distribution of heat at time $\Delta = 0$ is $\exp\left(cx^2\right)$ for some $c < 0$? If $\Delta > 1/(2c)$, the solution is given above. However, if $\Delta < 1/(2c)$, the distribution of heat at $\Delta = 0$ cannot go to zero at a rate of $\exp\left(cx^2\right)$ in the tails, even if all heat is initially concentrated at a single point; the heat will have diffused too much for these tails to go to zero at the specified rate.

Excessively thick or thin tails in the final condition are not the only problems that can cause power series to fail to converge. Consider the conditional moment:

$$f(\Delta, x) = E\left[\cos\left(cX_{t+\Delta}^2\right) \mid X_t = x\right] \tag{2.13}$$

for any real $c \neq 0$. (Without loss of generality, we can take $c > 0$.) The solution is given by:

$$f(\Delta, x) = \exp\left[-\frac{2x^2\Delta c^2}{1+4\Delta^2c^2}\right]\sqrt{\frac{1+\sqrt{1+4\Delta^2c^2}}{2}}\left[\begin{array}{c}\cos\left(\frac{x^2c}{1+4\Delta^2c^2}\right) \\ -\left(\frac{2\Delta c}{1+\sqrt{1+4\Delta^2c^2}}\right)\sin\left(\frac{x^2c}{1+4\Delta^2c^2}\right)\end{array}\right] \tag{2.14}$$

This solution is well-behaved for all real values of Δ; in this case, the quantities inside the square root signs are all positive and bounded away from zero. However, there are singularities in $f(\Delta, x)$ for imaginary values of Δ, at $\Delta = \pm i/(2c)$. The power series converges in the interval $\left(-\frac{1}{2c}, +\frac{1}{2c}\right)$ (and possibly also at the endpoints of the interval). Even though the desired function is perfectly well-behaved for all real values of Δ, singularities for complex values of Δ prevent convergence of a power series approximation for real values of Δ. Here, the problem is not that the final condition goes to either infinity or zero too rapidly, but that it oscillates too quickly in the tails.

Of course, other than their use as illustrative examples, there is little point in finding power series expansions to expressions that are known in closed-form. However, the problems encountered in the cases above can also occur when we do not know the solutions in closed form. Even though, in typical applications, we are interested only in the behavior of conditional moments (or asset prices) for positive time horizons, if the final (or payoff) condition has tails that are too thin or that oscillate too quickly, power series expansions can fail to converge even when the conditional moment or price is perfectly well-behaved for non-negative values of Δ. In the next section, we consider a parameterized family of time transformations that can remedy this problem, and that frequently extends convergence to all positive real values of $\Delta \in [0, +\infty)$, sometimes even uniformly.

3 Time Transformations

As noted in the preceding section, singularities for negative or complex values of Δ (i.e., values not of interest for most applications) can cause power series expansions in Δ to a function $f(\Delta, x)$ to fail to converge on the interval $\Delta \in [0, +\infty)$ (i.e., the values of Δ that are of interest). However, provided any singularities in $f(\Delta, x)$ are bounded away from $[0, +\infty)$, it is nonetheless possible to construct power series expansions that converge for all values of $\Delta \in [0, +\infty)$; in some cases, convergence on this interval can even be uniform. This goal can be accomplished by performing a change of the time variable, replacing Δ with some function $\tau(\Delta)$ and constructing a power series in τ instead of Δ. A power series converges within a circle in the complex plane; however, if the function $\tau(\Delta)$ is not affine in Δ, then this transformation effectively stretches time in some directions and compresses it in others, so that a circle in the complex plane of τ is some other shape in the complex plane of Δ. By appropriate choice of $\tau(\Delta)$, singularities in $f(\Delta, x)$ can effectively be moved away from the origin, whereas the interval $\Delta \in [0, +\infty)$ can be compressed, so that each point in this interval is closer to the origin. We now examine a transformation that accomplishes both of these goals, and that still allows computation of a power series by recursive computation of the coefficients.

3.1 The Basic Time Transform

We express the desired conditional moment or asset price as:

$$f(\Delta, x) = h(\tau, x) \tag{3.1}$$

where the function is given by:

$$\tau = \tau_k(\Delta) = 1 - \exp(-k\Delta) \tag{3.2}$$

for some range of values of Δ and some complex constant $k \neq 0$. Note that $\tau_k(\Delta)$ is a periodic function of Δ with period $2\pi i/k$, so this choice imposes strong restrictions on the functions $f(\Delta, x)$ which can be represented, if the range of Δ is not restricted. We assume either that $f(\Delta, x)$ has the necessary periodicity properties, or that the range of Δ is suitably restricted. The pricing PDE can then be rewritten in the following form:

$$k(1-\tau)\frac{\partial h}{\partial \tau}(\tau, x) = \sum_{i=1}^{N} [\mu(x)]_i \frac{\partial h}{\partial x_i}(\tau, x) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \left[\sigma^2(x)\right]_{ij} \frac{\partial^2 h}{\partial x_i \partial x_j}(\tau, x) - r(x)\, h(\tau, x) \tag{3.3}$$

This transformation has the property that:

$$\tau_k(0) = 0 \tag{3.4}$$

(regardless of the value of k), so the final condition is now expressed as:

$$h(0, x) = g(x) \tag{3.5}$$

We can now contemplate power series expansions of $h(\tau, x)$ in τ, rather than of $f(\Delta, x)$ in Δ:

$$h(\tau, x) = \sum_{i=0}^{\infty} b_i(x) \frac{\tau^i}{i!} \tag{3.6}$$

The coefficients can be found recursively, with the initial coefficient given by:

$$b_0(x) = g(x) \tag{3.7}$$

and subsequent coefficients $b_n(x)$ for $n \geq 1$ satisfying:

$$b_n(x) = \frac{1}{k} \sum_{i=1}^{N} [\mu(x)]_i \frac{\partial b_{n-1}}{\partial x_i}(x) + \frac{1}{2k} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[\sigma^2(x)\right]_{ij} \frac{\partial^2 b_{n-1}}{\partial x_i \partial x_j}(x) - \left[\frac{r(x)}{k} + 1 - n\right] b_{n-1}(x) \tag{3.8}$$

Thus a power series in τ can be calculated just as easily as a power series in Δ. It should be noted that the time-homogeneity of coefficients of the original pricing PDE is important; if these coefficients depended explicitly on time, then calculation of the coefficients by method similar to that of Equations 3.7 and 3.8 would be considerably complicated, and perhaps infeasible. However, we now examine the convergence properties of the power series in τ, and find they can be quite different than those of Δ for appropriate choice of k.

3.2 Small Circle Convergence

As discussed above, a power series converges within a circle in the complex plane, extending to the nearest singularity. Since the basic time transformation is non-affine, the region of convergence of a power series in τ (which is a circle in τ in the complex plane) is a non-circular shape in Δ in the complex plane; by appropriate choice of the parameter k of transformation, the power series in τ can often converge for positive real values of Δ which are larger than those values for which a power series in Δ converges. We now consider the requirements for convergence of the power series within a circle in $|\tau| < r$ with radius $0 < r < 1$; requirements for convergence in a circle with a larger radius are quite different, and are discussed in the next section. We note the inverse of the time transformation:

$$\Delta = \Delta_k(\tau) = \frac{\ln(1-\tau)}{-k} \tag{3.9}$$

The logarithm function, when viewed as a complex function of a complex argument, takes on multiple values; the logarithm function in Equation 3.9 is understood to take on values $-\pi < \mathrm{Im}[\ln(1-\tau)] \leq +\pi$. The inverse transformation also has a singular point at $\tau = 1$; however, this point lies outside the circle $|\tau| < r$, and therefore does not affect convergence of power series if $r < 1$. Within $|\tau| < r$ for $r < 1$, the inverse transformation is analytic. Consequently, the function $h(\tau, x)$:

$$h(\tau, x) = f(\Delta, x) \tag{3.10}$$

is analytic in τ at all points $\Delta = \Delta_k(\tau)$, $|\tau| < r$, where $f(\Delta, x)$ is analytic in Δ. The following theorem characterizes a region of Δ in which analyticity guarantees the convergence of the power series in τ.

Theorem 1. *Let $k \neq 0$ be a complex constant, and let $0 < r < 1$. If $f(\Delta, x)$ is defined and analytic in Δ in the region where:*

$$-\arccos\sqrt{1-r^2} \quad < \quad \mathrm{Im}(k\Delta) < \arccos\sqrt{1-r^2} \tag{3.11}$$

$$-\ln\left(\frac{\cos\left[\mathrm{Im}(k\Delta)\right]}{+\sqrt{\cos^2\left[\mathrm{Im}(k\Delta)\right]-(1-r^2)}}\right) \quad < \quad \mathrm{Re}(k\Delta) < -\ln\left(\frac{\cos\left[\mathrm{Im}(k\Delta)\right]}{-\sqrt{\cos^2\left[\mathrm{Im}(k\Delta)\right]-(1-r^2)}}\right) \tag{3.12}$$

then the function $h(\tau, x) = f(\Delta, x)$ (where $\Delta = \Delta_k(\tau)$ is defined by Equation 3.9) is analytic in τ in the region $|\tau| < r$. Conversely, if $h(\tau, x)$ is defined and analytic in τ in the region $|\tau| < r$, then $f(\Delta, x) = h(\tau, x)$ (where $\tau = \tau_k(\Delta)$ is defined by Equation 3.2) is analytic in Δ in the region indicated by Equations 3.11 and 3.12.

Proof: See appendix.

A consequence of Theorem 1 is that, even when a power series expansion in Δ to $f(\Delta, x)$ converges only in the region $|\Delta| < c$, and in particular, for positive real values $0 < \Delta < c$, it is often possible that a power series approximation in τ to $h(\tau, x) = f(\Delta, x)$ converges for values of τ that correspond to positive real values of Δ which are greater than c. Figures 1 and 2 show the graph of $\Delta_k(\tau)$ for circles $|\tau| = r$ with various radii $0 < r < 1$, and for $k = 1$ and $k = 2$, respectively.

As shown in Figure 1, when the parameter k of the transformation given by Equation 3.2 is a positive real number, a circle in τ maps to an elongated shape in Δ, with the direction of elongation to the right. Consequently, a power series expansion in τ can converge for larger positive values of Δ than a power series expansion in Δ. For example, if a function has a singularity at $\Delta = -0.5$ (and at no other point), a power series in Δ converges for positive real values $0 < \Delta < 0.5$. However, a power series in τ, given by Equation 3.2 with $k = 1$, converges at least within the region inside the $r = 0.3$ circle, which includes positive real values of Δ up to approximately 0.7935. In fact, the largest circle in τ which does not include $\Delta = -0.5$ has radius of approximately 0.4208, and includes positive real values of Δ up to approximately 1.0462. The use of a time transformation therefore increases the interval of convergence on the positive real line.

Figure 1 shows circles in τ mapped to Δ when the parameter of the time transformation is $k = 1$. When the parameter of transformation is larger, the elongation of the circles in τ (when mapped to Δ) is more extreme. Figure 2 shows circles in τ mapped to Δ when the parameter of the time transformation is $k = 2$. Note, for example, that a singularity at $\Delta = -0.5$, which prevents convergence of series in τ for $\Delta > 1.0462$ when $k = 1$, allows convergence of series in τ for values of Δ up to at least 2.5; in fact, the convergence is for arbitrarily large positive values of Δ, since every such value of Δ is included in some circle with radius $r < 1$,

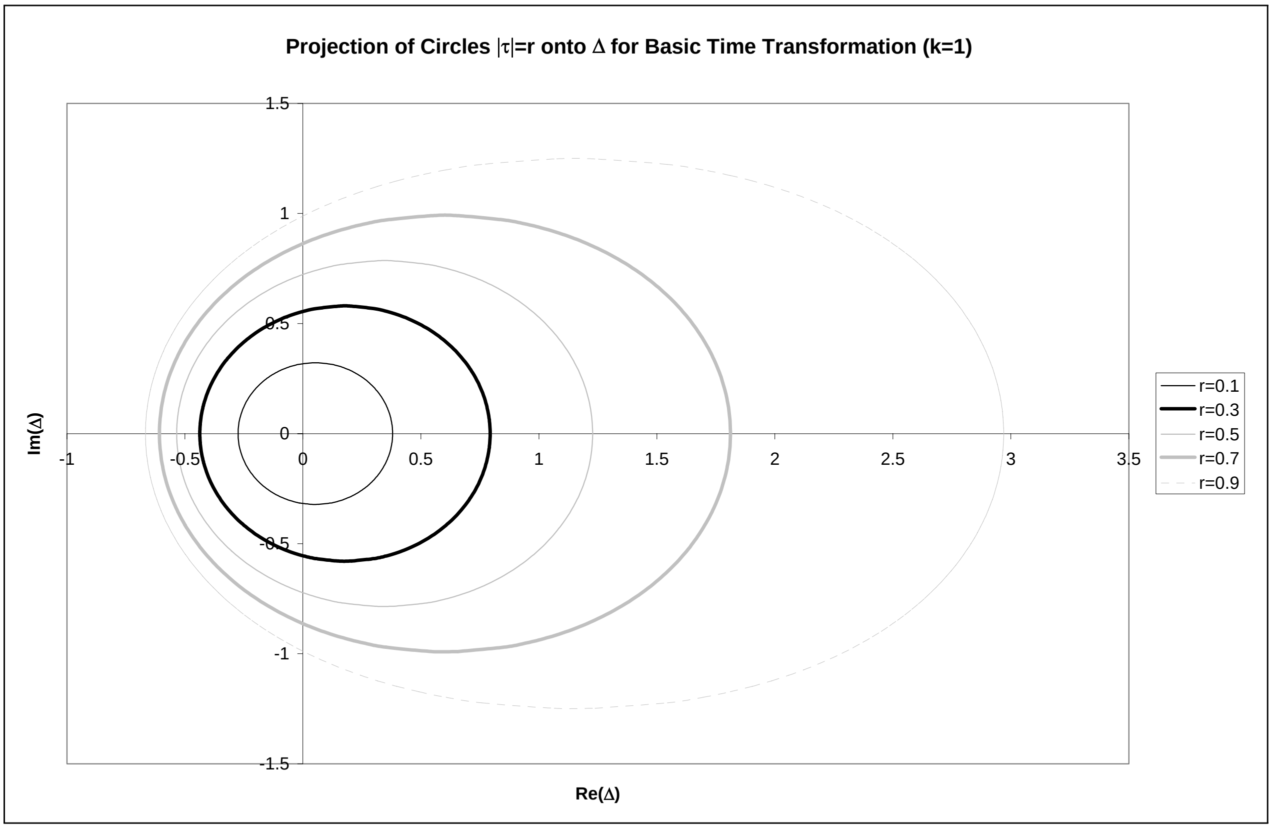

Figure 1: This figure shows the position of circles $|\tau| = r$ with radii $0 < r < 1$ as a function of Δ, where the parameter of transformation of Equation 3.2 is $k = 1$. As shown, the projections of the circles in τ onto Δ are elongated shapes which extend farther in the positive real direction than in the negative real direction. The direction of elongation is the same as the direction of k in the complex plane. If a function is analytic in Δ at every point inside one of the circles in τ, then a power series in τ converges within that region.

but no such circle includes the point $\Delta = -0.5$.

The following corollary provides conditions that guarantee convergence of a power series in τ corresponding to arbitrarily large positive values of Δ.

Corollary 1. *Let $k \neq 0$ be a complex constant. If $f(\Delta, x)$ is defined and analytic in Δ in the region where:*

$$-\frac{\pi}{2} < \mathrm{Im}(k\Delta) < \frac{\pi}{2} \tag{3.13}$$

$$\mathrm{Re}(k\Delta) > -\ln(2) - \ln(\cos[\mathrm{Im}(k\Delta)]) \tag{3.14}$$

then the function $h(\tau, x) = f(\Delta, x)$ (where $\Delta = \Delta_k(\tau)$ is defined by Equation 3.9) is analytic in τ in the region $|\tau| < 1$. Conversely, if $h(\tau, x)$ is defined and analytic in τ in the region $|\tau| < 1$, then $f(\Delta, x) = h(\tau, x)$ (where $\tau = \tau_k(\Delta)$ is defined by Equation 3.2) is analytic in Δ in the region indicated by Equations 3.13 and 3.14.

Proof: See appendix.

Note that, in contrast to Theorem 1, which places both upper and lower bounds on $\mathrm{Re}(k\Delta)$, Corollary 1 imposes only a lower bound. As shown in Figure 3, the unit circle $|\tau| = 1$ maps to an open shape in Δ; if the parameter of the time transformation k is real and positive, then the opening is towards the right. For larger values of k, the circle (mapped to τ) follows the positive real axis in Δ more closely; provided $f(\Delta, x)$ has no singularities in the neighborhood of the positive real axis, convergence for arbitrarily large positive real values of Δ can be guaranteed by choosing a sufficiently large value of k.

At this point, it is worth revisiting some of the examples of Section 2; although the solutions are known in closed form, they nonetheless provide illustrative examples of phenomena which also occur when the solutions are not known in closed form. The function $f(\Delta, x)$ defined in Equation 2.12 has a singularity at $\Delta = 1/(2c)$. If c is positive, the basic time transformation cannot extend the interval of convergence to positive real values $\Delta \geq 1/(2c)$. However, if c is negative, the singularity occurs at a negative value of Δ, but nonetheless prevents convergence of a power series expansion of $f(\Delta, x)$ in Δ for positive real values $\Delta \geq 1/(2c)$. However, applying the basic time transformation of Equation 3.2 with a positive real value of k extends the interval of convergence to larger positive real values of Δ. In particular, if the parameter of transformation k is chosen so that $k \geq -2c\ln 2$, then the conditions of Corollary 1 are satisfied, and a power series expansion of $f(\Delta, x)$ in τ converges for all positive real values of Δ. Similarly, the function $f(\Delta, x)$ given by Equation 2.14 has singularities at $\Delta = \pm i/(2c)$, and a power series expansion of $f(\Delta, x)$ in Δ converges only in the interval $\Delta \in (-1/(2c), +1/(2c))$ for $(c > 0)$. However, if the parameter of transformation k is chosen so that $k \geq 2c\pi/3$, then once more, the conditions of Corollary 1 are satisfied, and a power series expansion of $f(\Delta, x)$ in τ converges for all positive real values of Δ.

Finally, we note that, when we apply Theorem 1 and Corollary 1 in Sections 4 and 5, we generally find it advantageous to take $f(\Delta, x)$ as given, derive its analyticity properties, and then find the function $h(\tau, x)$ with the desired analyticity properties, rather then taking $h(\tau, x)$ as given and then deriving $f(\Delta, x)$. However, if we take the latter approach, it is possible to replace Equation 3.11 by:

$$\sqrt{1 - r} < \cos[\mathrm{Im}(k\Delta)] \tag{3.15}$$

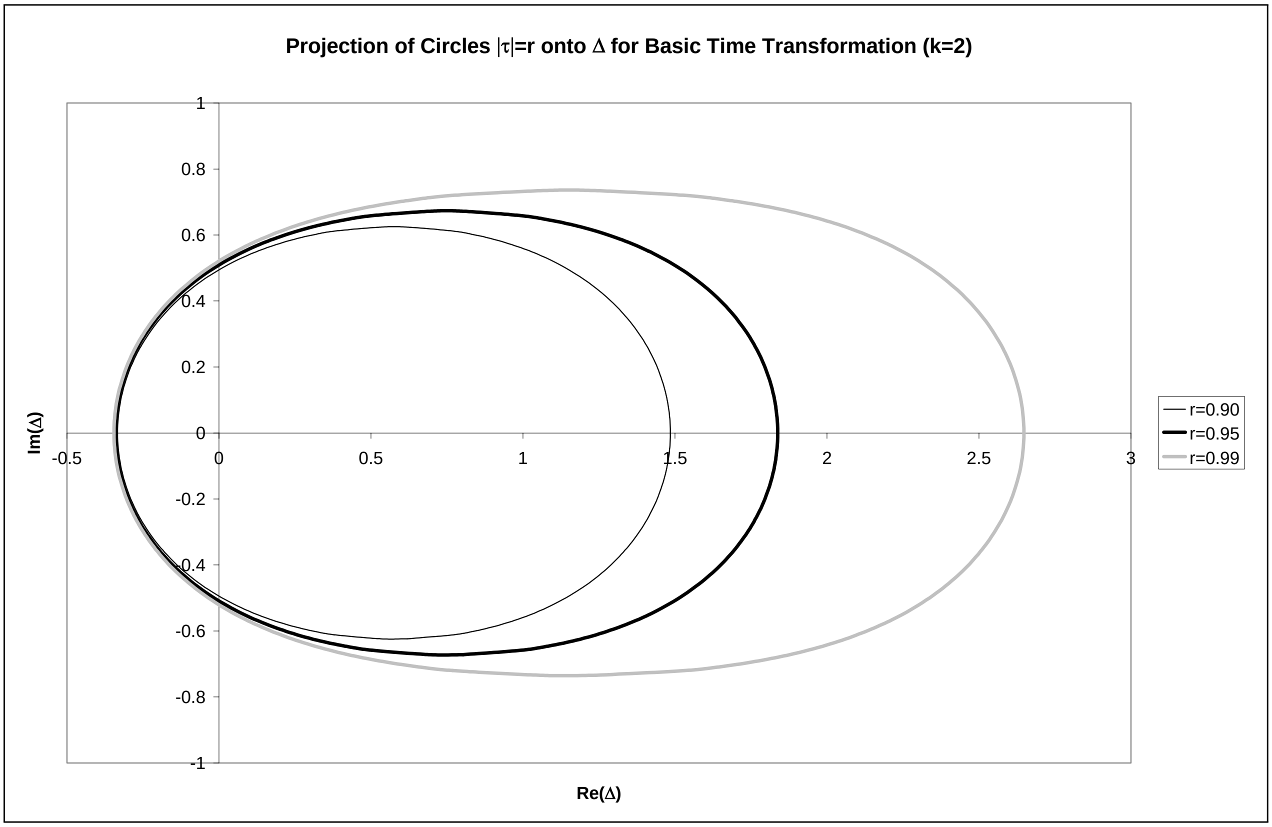

Figure 2: This figure shows the position of circles $|\tau| = r$ with radii $0 < r < 1$ as a function of Δ, where the parameter of transformation of Equation 3.2 is $k = 2$. As shown, the projections of the circles in τ onto Δ are elongated shapes which extend farther in the positive real direction than in the negative real direction. The direction of elongation is the same as the direction of k in the complex plane. If a function is analytic in Δ at every point inside one of the circles in τ, then a power series in τ converges within that region. Note that, with parameter of transformation $k = 2$, the circles in τ are more elongated and follow the positive real axis more closely than in the $k = 1$ case. A power series in τ therefore always converges for at least as large a range of positive real values of Δ in the $k = 2$ case as in the $k = 1$ case.

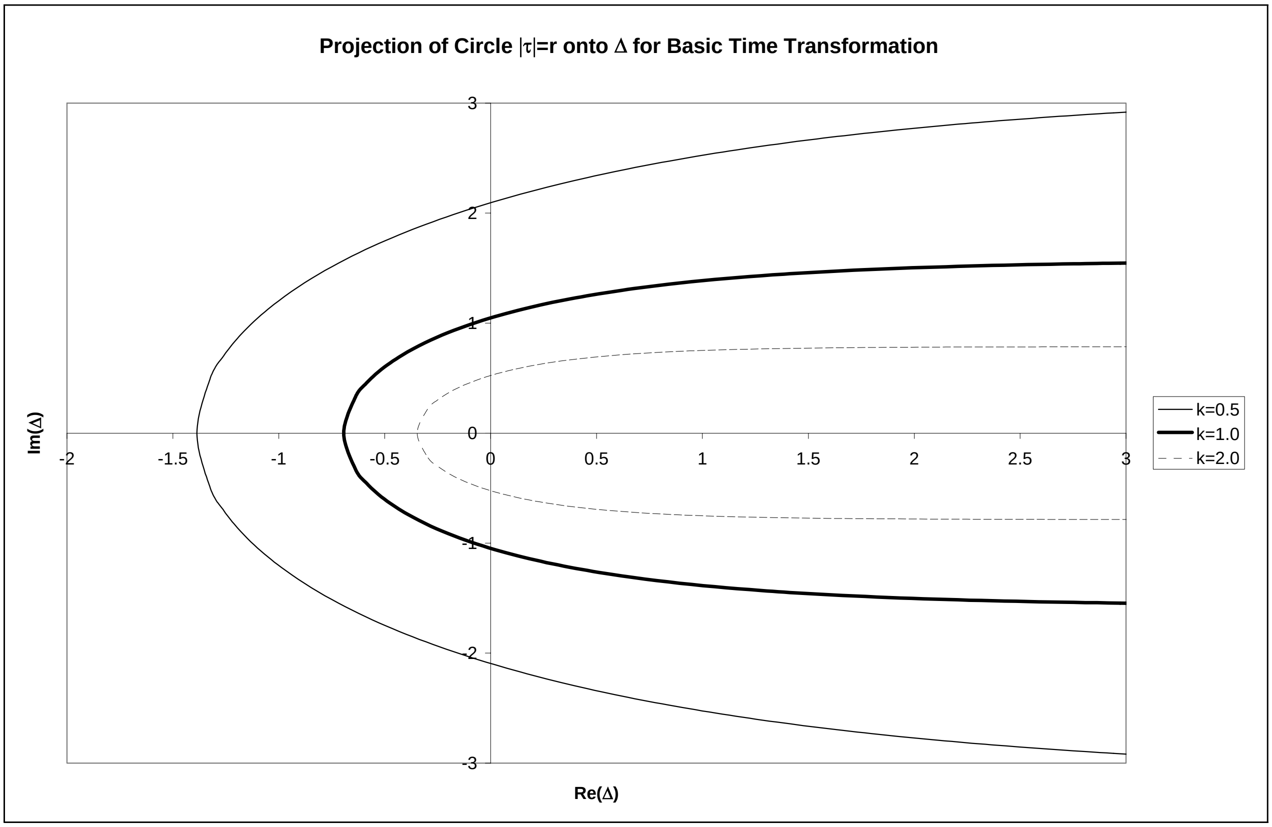

Figure 3: This figure shows the position of the circles $|\tau| = r$ as a function of Δ, for different values of the parameter of transformation of Equation 3.2. As shown, the unit circle projections onto Δ are elongated shapes opening up towards the right (i.e., towards large positive real values) in the complex plane; the direction of the opening is the same as the direction of k in the complex plane. If a function is analytic in Δ at every point inside of the circles in τ (i.e., to the right of the the corresponding shapes in the figure), then a power series in τ converges within that region.

and Equation 3.13 by:

$$0 < \cos\left[\mathrm{Im}\left(k\Delta\right)\right] \tag{3.16}$$

In this case, the function $f(\Delta, x)$ is defined for a wider range of values of Δ, and is a periodic function of Δ with period $2\pi i/k$. The projection of $|\tau| < r$ for $r \leq 1$ onto Δ then consists of infinitely many of the elongated shapes shown in Figures 1, 2, or 3, each spaced $2\pi i/k$ above or below the next. Note, however, that the function $f(\Delta, x)$, defined within one of these regions, may have an analytic extension that is not periodic, and therefore is not equal to the value of $f(\Delta, x)$ defined within the other elongated regions.

Although convergence for arbitrarily large values of Δ can sometimes be established (as in the two examples studied in this section), such convergence will, in general, not be uniform on $\Delta \in [0, +\infty)$. In some cases, uniform convergence can be established, but the conditions for such convergence are considerably more complicated. We examine this situation in the next section.

3.3 Large Circle Convergence

The previous section considered the convergence properties of power series in τ for circles $|\tau| < r$ with $0 < r \leq 1$, and establishes sufficient conditions for convergence on intervals such as $\Delta \in [0, +\infty)$. However, such convergence is not, in general, uniform. Uniform convergence can sometimes be established by considering circles $|\tau| < r$ with $r > 1$, but the analysis is much more complicated. If the radius of the circle in τ is larger than 1, then the circle includes the point $\tau = 1$, which does not correspond to any value of Δ. The following theorem establishes conditions relating analyticity in τ to analyticity in Δ for circles (in τ) with radius $r > 1$; note that the conditions are quite different than those of Theorem 1 and Corollary 1. Unlike these results, which allow us to begin with either of the functions $f(\Delta, x)$ or $h(\tau, x)$ and define and characterize the other function, the following theorem assumes $h(\tau, x)$ is given, and derives the properties of $f(\Delta, x)$.

Theorem 2. *Let $k \neq 0$ be a complex constant, and let $r > 1$. If $h(\tau, x)$ is defined and analytic within the circle $|\tau| < r$, with $r > 1$, then $f(\Delta, x) = h(\tau, x)$ (where $\tau = \tau_k(\Delta)$ is defined by Equation 3.2) is defined and analytic in the region where:*

$$\mathrm{Re}\left(k\Delta\right) > -\ln\left(\cos\left[\mathrm{Im}\left(k\Delta\right)\right] + \sqrt{\cos^2\left[\mathrm{Im}\left(k\Delta\right)\right] + r^2 - 1}\right) \tag{3.17}$$

Furthermore, $f(\Delta, x)$ is periodic in this region with period $2\pi i/k$, and tends to a limit as $\mathrm{Re}(k\Delta) \to +\infty$.

Proof: See appendix.

When Theorem 2 is applied in Sections 4 and 5, it is the definition, rather than the analyticity, of the function $h(\tau, x)$ that is important. Since, by assumption, the function $h(\tau, x)$ is analytic within $|\tau| < r$ for some $r > 1$, a power series approximation in τ to $h(\tau, x)$ therefore converges uniformly in any circle $|\tau| \leq s$ with $s < r$, and in particular for some s such that $1 < s < r$. For positive real values of k, from Equation 3.17, the entire positive real axis $\Delta \in [0, +\infty)$ maps to values $|\tau_k(\Delta)| \leq s$, regardless of the value of k. Thus, if a function $f(\Delta, x)$ can be constructed from a function $h(\tau, x)$ that satisfies the conditions of Theorem 2 for a positive real value of k, a power series in $\tau_k(\Delta)$ converges uniformly for all $\Delta \in [0, +\infty)$. In Sections 4 and

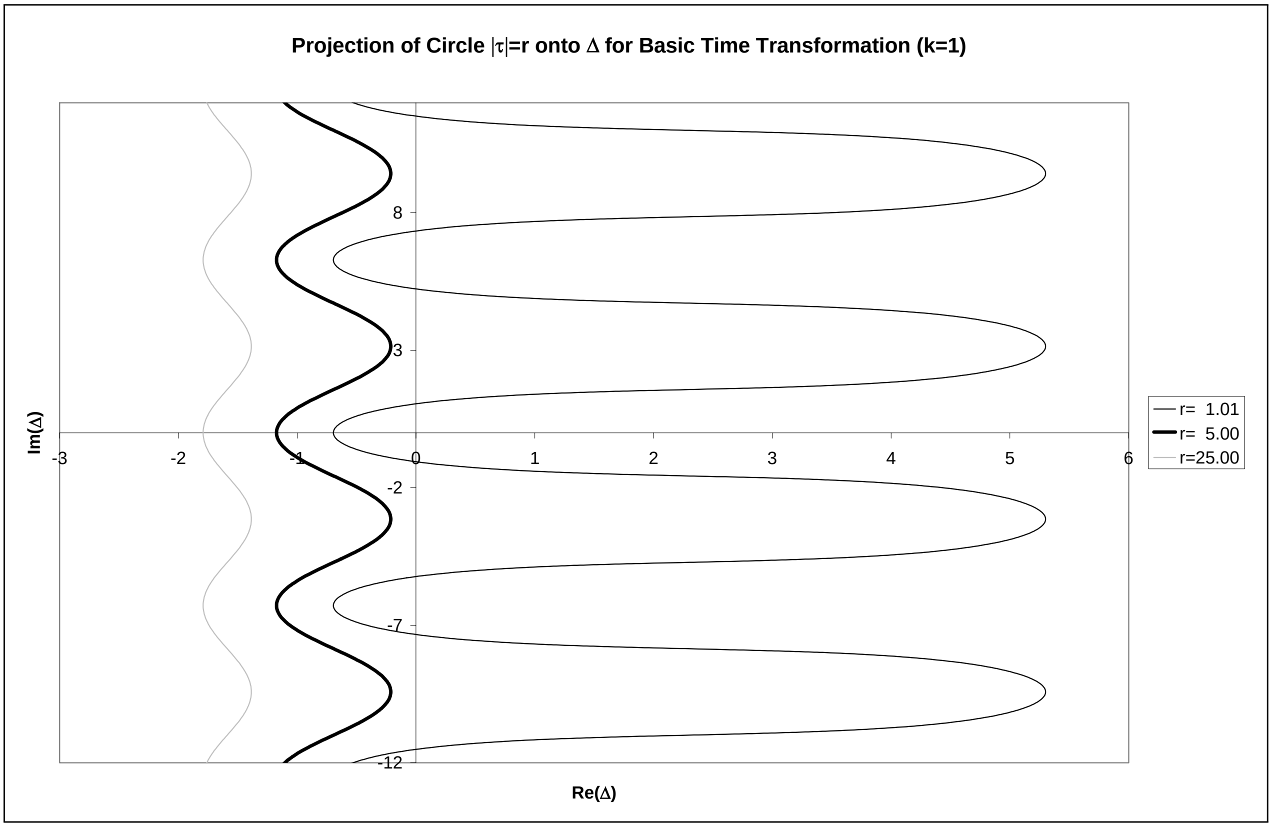

Figure 4: This figure shows the position of circles $|\tau| = r$ as a function of Δ, where the parameter of the transformation of Equation 3.2 is $k = 1$. As shown, the projections of the circles in τ onto Δ are periodic functions of $\mathrm{Im}\,(k\Delta)$. Points to the right of the curves are inside the corresponding circles in τ, and points to the left are outside the same circles. If a function is analytic within the indicated circle in τ, then it is defined as a function of Δ in the indicated region. Note that all the circles shown include the interval $\Delta \in [0, +\infty)$.

5, however, we must assume strong conditions on the coefficients and final condition of the PDE that gives rise to $f\,(\Delta, x)$ to ensure applicability of Theorem 2; much weaker conditions are needed for applicability of Theorem 1 and Corollary 1.

The region described by Equation 3.17 is quite different than that described in either Theorem 1 or Corollary 1; in Figure 4, we examine this region for circles of various radii with $k = 1$. Since all three "circles" shown in Figure 4 have radius greater than one, if the goal is establishing uniform convergence on $\Delta \in [0, +\infty)$, the $r = 1.01$ circle is sufficient; establishing the necessary conditions for analyticity of $h\,(\tau, x)$ to ensure existence and analyticity of $f\,(\Delta, x)$, within, for example, the $r = 5$ circle, while possibly useful for some other applications, is of no use in establishing convergence of $f\,(\Delta, x)$ on the positive real line.

The role of the parameter of transformation k is also fundamentally different as soon as we discuss circles with radius greater than 1. For circles of radius less than 1, larger values of k decrease the projection of the circle in τ onto Δ; as shown in Figure 5 for circles of radius 1.01, this is not necessarily the case for circles

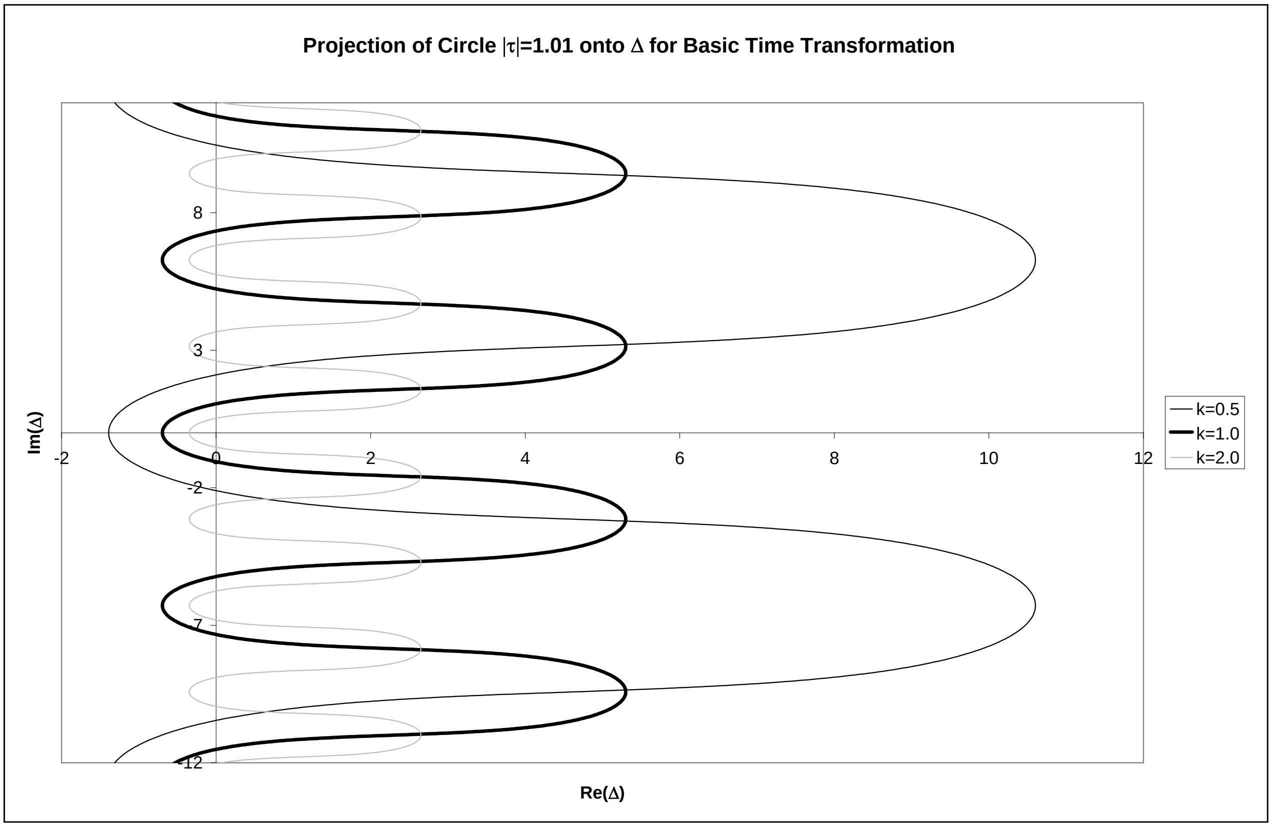

Figure 5: This figure shows the position of circles $|\tau| = 1.01$ as a function of Δ, for different values of the parameter of the transformation of Equation 3.2. As shown, the projections of the circles in τ onto Δ are periodic functions of $\mathrm{Im}\,(k\Delta)$. Points to the right of the curves are inside the corresponding circles in τ, and points to the left are outside the same circles. If a function is analytic within the indicated circle in τ, then it is defined as a function of Δ in the indicated region. Note that all the circles shown include the interval $\Delta \in [0, +\infty)$.

with radius $r > 1$. There is no circle for which the region of definition and analyticity of $f(\Delta, x)$ is a strict subset of the corresponding region for another circle. Furthermore, Theorem 2 produces $f(\Delta, x)$ functions which are periodic; if the goal is to find a function $h(\tau, x)$ that, after application of Theorem 2, generates a given $f(\Delta, x)$, it is clear that the possible values of k are several restricted by the periodicity of $f(\Delta, x)$; if $f(\Delta, x)$ is not periodic, then there is no value of k such that an appropriate $h(\tau, x)$ can be found.

The examples of Section 2, described in Equations 2.12 and 2.14, are not periodic in Δ. However, moments of an Ornstein-Uhlenbeck process are. Consider:

$$\mu(x) = -x \tag{3.18}$$

$$\sigma(x) = 1 \tag{3.19}$$

$$r(x) = 0 \tag{3.20}$$

with final condition:

$$g(x) = x \tag{3.21}$$

The conditional moment is given by:

$$f(\Delta, x) = xe^{-\Delta} \tag{3.22}$$

This function can be written as:

$$h(\tau, x) = f(\Delta_1(\tau), x) = x(1-\tau) \tag{3.23}$$

The function $h(\tau, x)$ is clearly analytic in τ within a circle $|\tau| < r$ for some $r > 1$ (in fact, $h(\tau, x)$ is entire), so a power series in τ converges uniformly for all $|\tau| \leq 1$, which includes the interval $\Delta \in [0, +\infty)$. The same result can be had by choosing any $\tau = \tau_{1/m}(\Delta)$ for any integer $m > 0$:

$$h(\tau, x) = f(\Delta_1(\tau), x) = x(1-\tau)^m \tag{3.24}$$

A power series expansion in τ still converges uniformly on an interval including $\Delta \in [0, +\infty)$. Note, however, that the function $h(\tau, x)$ is not analytic in τ at $\tau = 1$ for any other value of k, so for such values, a power series expansion in τ can only be guaranteed to converge within the circle $|\tau| < 1$.

In the preceding example, uniform convergence on $\Delta \in [0, +\infty)$ is achieved if the parameter of transformation is $k = 1/m$ for some integer $m > 0$. However, the situation can change for different final conditions. Consider the conditional expectation (where the coefficients of the PDE are given by Equations 3.18 through 3.20) of:

$$g(x) = e^{-cx^2} \tag{3.25}$$

We consider real values of $c > -1$, in which case the solution is given by:

$$f(\Delta, x) = \frac{\exp\left(\frac{cx^2}{c-(c+1)\exp(2\Delta)}\right)}{\sqrt{1 + c\left(1 - \exp\left(-2\Delta\right)\right)}} \tag{3.26}$$

If we choose $k = 2/n$ for any integer $n > 0$, then this solution can be written as:

$$h(\tau, x) = f\left(\Delta_{2/n}(\tau), x\right) = \frac{\exp\left(\frac{cx^2(1-\tau)^n}{c(1-\tau)^n-(c+1)}\right)}{\sqrt{1 + c\left[1 - (1-\tau)^n\right]}} \tag{3.27}$$

In the specific case $k = 2$, we have:

$$h(\tau, x) = f(\Delta_2(\tau), x) = \frac{\exp\left(\frac{cx^2(\tau-1)}{c\tau+1}\right)}{\sqrt{1 + c\tau}}$$

This function has a singularity at $\tau = -1/c$. If $c < 1$, then this singularity lies outside the unit circle in τ, and the conditions of Theorem 2 are satisfied for some $r > 1$. In other words, provided the tails of the final condition do not go to zero too quickly, a power series approximation to $f(\Delta, x)$ in $\tau = \tau_2(\Delta)$ converges uniformly on $\Delta \in [0, +\infty)$.

However, in the case $c > 1$, the singularity at $\tau = -1/c$ lies within the unit circle in τ, and the conditions of Theorem 2 are not satisfied for $k = 2$. Choosing $k = 2/n$ for some $n > 1$ does not improve the situation. Consequently, if the tails of the final condition go to zero too quickly, the conditions of Theorem 2 cannot be satisfied for any value of k, and the basic time transformation can not result in uniform convergence on $\Delta \in [0, +\infty)$ (although the weaker conditions of Corollary 1 do result in non-uniform convergence on this interval).

However, in some cases, even when the conditions of Theorem 2 are not satisfied for any value of k, it may nonetheless still be possible to use the basic time transformation iteratively, with uniform convergence on $\Delta \in [0, +\infty)$ resulting. The next section explores this possibility.

3.4 Compound Time Transform

For positive real values of k, the basic time transformation $\tau = \tau_k(\Delta)$ maps the positive real axis $\Delta \in [0, +\infty)$ to the interval $\tau \in [0, 1)$. Uniform convergence of a power series in τ on $\Delta \in [0, +\infty)$ requires analyticity of the function being approximated in a circle $|\tau| < r$ for some $r > 1$. It may well be the case that the the function $h(\tau, x)$ is analytic in some neighborhood of $\tau \in [0, 1]$, but not within the circle $|\tau|$. In this case, on possible approach is to apply Theorem 1 to τ instead of Δ, replacing τ by a non-affine transformation $\gamma = \tau_k(\tau)$. This approach is effective in dealing with the example of the previous section when $c > 1$. The singularity then lies at $\tau = -1/c > -1$, which is within the unit circle in τ. But if we can find some $k > 0$ and $0 < r < 1$ such that the interval:

$$-\ln\left(1 + \sqrt{r}\right) < \mathrm{Re}\,(k\tau) < -\ln\left(1 - \sqrt{r}\right) \tag{3.28}$$

includes $\tau = 1$ but not $\tau = -1/c.$, the conditions of Theorem 1 are satisfied, and a power series in γ converges uniformly for $\tau \in [0, 1)$, and therefore for $\Delta \in [0, +\infty)$. Choosing $c\ln(1 + \sqrt{r}) < k < -\ln(1 - \sqrt{r})$ satisfies both criteria, so a power series in γ has the desired uniform convergence properties. The combined effects of both time transforms can be written as:

$$\gamma = \tau_{k_1,k_2}(\Delta) = \tau_{k_2,}(\tau_{k_1}(\Delta)) = 1 - \exp\left(-k_2\left[1 - \exp\left(-k_1\Delta\right)\right]\right) \tag{3.29}$$

Although the compound transform described above does mean the power series expansion of the desired function in γ converges uniformly on $\Delta \in [0, +\infty)$, calculating the coefficients of the transformation is not trivial; Equations 3.7 and 3.8 may not be applicable, since the coefficients of the PDE satisfied by $h(\tau, x)$ are, in general, not time-homogeneous. However, as we see in Sections 4 and 5, there does exist an important set of problems for which a change of independent variable, in addition to the change of time variable, establishes time homogeneity in the coefficients once again. For these problems, the compound time transform can be a very useful tool in establishing uniform convergence on $\Delta \in [0, +\infty)$.

3.5 State Dependency

Throughout the discussion in the preceding sections, we have considered the problem of approximating a function $f(\Delta, x)$ by series expansion in Δ while holding the value of x fixed. The power series of $h(\tau, x)$ in τ converges within a circle in τ, which, for a given value of k, contains a portion of the positive real axis $\Delta \in [0, +\infty)$, and possibly the entire positive real axis; in some cases, convergence on the interval $\Delta \in [0, +\infty)$ is uniform. It is not obvious that it will always be possible to choose a single value of k that guarantees convergence on some interval $\Delta \in [0, T)$ for all x. For example, consider the conditional mean of the deterministic process:

$$dX_t = -X_t^2 dt \tag{3.30}$$

The corresponding coefficients of Equation 2.3 are given by:

$$\mu(x) = -x^2 \tag{3.31}$$

$$\sigma(x) = 0 \tag{3.32}$$

$$r(x) = 0 \tag{3.33}$$

with final condition:

$$g(x) = x \tag{3.34}$$

The closed form solution to this problem is:

$$f(\Delta, x) = \frac{x}{1 + x\Delta} \tag{3.35}$$

This function has a singularity at:

$$\Delta = -\frac{1}{x} \tag{3.36}$$

Since the initial value is positive, this singularity always lies on the negative real axis. However, for very large initial values x, the singularity can occur arbitrarily close to the origin. For a fixed value of x, the conditions of Corollary 1 apply, and the power series in τ converges for all $\Delta \in [0, +\infty)$ for any sufficiently large k. However, there is no single value of k that works for all values of x; for a given value of k, convergence on $\Delta \in [0, +\infty)$ occurs if there is no singularity on the negative real line within a given distance of the origin. By increasing x, the singularity can be moved arbitrarily close to the origin, and series expansions based on a particular value of k will fail to converge. The value of k in the transformation $\tau_k(\Delta)$ must therefore depend on x if it is to result in convergence on $(\Delta, x) \in [0, +\infty) \times (0, +\infty)$. This phenomenon is shown graphically in Figure 6, which identifies combinations of Δ and x for which power series (after exponential transformation) converge for different values of k.

In the specific cases of non-degenerate stochastic processes we examine in the next two sections, this phenomenon does not occur; that is, for a given value of k, the circle of convergence in Δ does not depend on

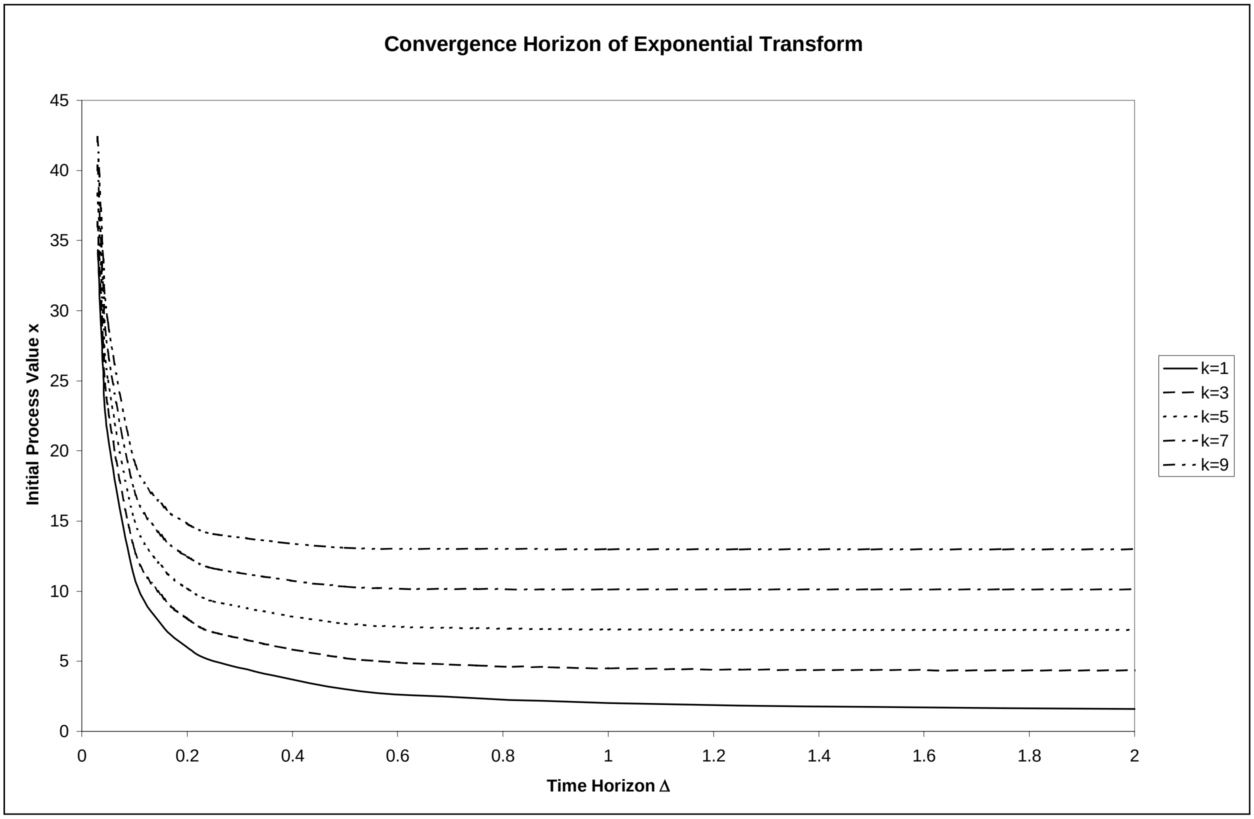

Figure 6: This figures identifies the values of Δ and x for which power series of the deterministic process described in Equations 3.31 through 3.34 (after exponential transformation) converge for various values of k. If the point corresponding to given values of Δ and x (note that both values are positive and real) is located below or to the left of a particular curve on the graph, the power series converges for that value of k; if it is above or to the right, the power series diverges. As shown, there is a tradeoff between the two variables; for a given value of k, the power series converges for large values of Δ only when x is small, and for large values of x only when Δ is small.

the value of x. However, whether there exist non-degenerate cases in which the circle of convergence of Δ is state-dependent remains an open issue.

4 Characterizing Analyticity - Scalar Diffusions

We now consider scalar processes, and examine the problem of characterizing final conditions which yield analytic solutions to the Feynman-Kac (or Chapman-Kolmogorov backward) equation. Analysis of these PDEs is sometimes greatly simplified by the use of transformations of the independent, dependent, and time variables. We demonstrate that, under only mild technical restrictions on the coefficients of the pricing PDE, there exists a large family of final conditions, such that prices are analytic in the time variable not only in a neighborhood of the origin, but in the entire complex plane. We proceed to examine several specific cases in which the family of final conditions that generate analytic (or entire) solutions in the time variable can be characterized explicitly.

4.1 Change of Variables

Change of independent variable is a technique frequently used to simplify analysis of a diffusion process (or, equivalently, a parabolic partial differential equation). Less used in the finance and economic literature are changes of time variable or dependent variable, although the latter technique has been used in the partial differential equation literature; see, for example, Colton (1979). By the use of such transformations, solution of the general Feynman-Kac problem can often be reduced to solution of a special case, although, in general, these transforms cannot easily be applied to multivariate diffusions. Consider the problem of finding the expectation:

$$f(\Delta, x) = E\left[g(X_{t+\Delta}) \exp\left(-\int_t^{t+\Delta} r(X_u)\, du\right) \mid X_t = x\right] \tag{4.1}$$

where X_t is a diffusion process which solves:

$$X_{t+\Delta} = X_t + \int_t^{t+\Delta} \mu(X_u)\, du + \int_t^{t+\Delta} \sigma(X_u)\, dW_u \tag{4.2}$$

and W_t is a Brownian motion. Under technical regularity conditions, this problem is equivalent to the problem of solving the partial differential equation:

$$\frac{\partial f}{\partial \Delta}(\Delta, x) = \mu(x) \frac{\partial f}{\partial x}(\Delta, x) + \frac{\sigma^2(x)}{2} \frac{\partial^2 f}{\partial x^2}(\Delta, x) - r(x) f(\Delta, x) \tag{4.3}$$

with final condition:

$$f(0, x) = g(x) \tag{4.4}$$

We seek solutions to this equation for all $x \in (a, b)$ where a and b are the boundaries of the diffusion process (with $a = -\infty$ or $b = +\infty$ or both possible) and $\Delta \in [0, T)$ for some $T > 0$ (with $T = +\infty$ possible). We

	Coefficients after Transformation		
Transformation	μ	σ	r
Scale function	0	$\sigma(x)$	N/A
Aït-Sahalia (2002)	$\mu(x)$	1	N/A
Colton (1979)	0	1	$r(x)$

Table 1: Scalar Diffusion Transformations

This table shows several methods of transformation designed to simplify scalar diffusions. In all cases, the transformed diffusion is characterized by a single function.

require $\sigma(x) \neq 0$ for all $x \in (a, b)$; as the PDE is motivated by a diffusion process, it will usually be the case that $\mu(x)$ and $\sigma(x)$ are chosen so that the boundaries a and b cannot be reached in finite time, but it is quite possible to analyze the PDE problem without imposing such restrictions. In general, there will be multiple solutions to the PDE problem, even with the given boundary condition, and at most one of these solutions is also the solution to the probabilistic problem. However, there can be at most one solution which is analytic in the time variable, and, if it exists, this solution is the one that is needed.

Table 1 shows some changes in variables which have been used to simplify stochastic processes and/or partial differential equations. The first two transformations shown are both changes of independent variable, and are usually used in cases where $r(x) = 0$, although they can also be used for non-zero $r(x)$. The scale transformation (see, for example, Karlin and Taylor (1981)) is as follows:

$$y = \int^{x} \exp\left(-\int^{v} \frac{2\mu(u)}{\sigma^2(u)} du\right) dv \tag{4.5}$$

When the PDE is expressed with y as the independent variable instead of x, the coefficient on the first spatial derivative is zero; if the $r(x)$ function is already zero, then only the second term on the right-hand side of the pricing PDE is non-zero after this change of independent variable.

A different change of independent variable is used by Aït-Sahalia (2001), who sets:

$$y = \int^{x} \frac{du}{\sigma(u)} \tag{4.6}$$

In the resultant PDE with y instead of x as independent variable, the coefficient of the second derivative is 1/2. In this case, provided the $r(x)$ function is already zero, all information in the coefficients of the pricing PDE is now contained in the drift coefficient. Both of these transformations are always well-defined. They are also monotonic, and therefore invertible, although in practice, it may be difficult to evaluate the integrals explicitly.

Colton (1979), who transforms both the dependent and independent variables, is able to eliminate the first term on the right-hand side of the PDE and normalize the second term simultaneously. We employ this latter technique, using the transforms:

$$y = \int^x \frac{du}{\sigma(u)} \tag{4.7}$$

$$f(\Delta, x) = \exp\left(-\int^x \left[\frac{\mu(u)}{\sigma^2(u)} - \frac{\sigma'(u)}{2\sigma(u)}\right] du\right) h(\Delta, y) \tag{4.8}$$

(Note that the lower limits of the integrals are unspecified, so these expressions really describe a family of transforms.)

The differential equation is now:

$$\frac{\partial h}{\partial \Delta} = \frac{1}{2}\frac{\partial^2 h}{\partial y^2} + \left[-\frac{\mu^2(x)}{2\sigma^2(x)} - \frac{\mu'(x)}{2} + \frac{\mu(x)\sigma'(x)}{\sigma(x)} - \frac{\sigma'(x)\sigma'(x)}{8} + \frac{\sigma''(x)\sigma(x)}{4} - r(x)\right] h \tag{4.9}$$

$$= \frac{1}{2}\frac{\partial^2 h}{\partial y^2} - r_h(y) h \tag{4.10}$$

with final condition:

$$h(0, y) = g_h(y) \equiv \exp\left(\int^x \left[\frac{\mu(u)}{\sigma^2(u)} - \frac{\sigma'(u)}{2\sigma(u)}\right] du\right) g(x) \tag{4.11}$$

Note that, provided the diffusion coefficient is bounded away from zero on the interior of the state space, these transforms are always well-defined (although in some cases we may not be able to evaluate the integral explicitly). Positivity of $\sigma(x)$ means that y is a strictly monotonic function of x, and can therefore be inverted; the process Y_t, defined by applying the change of independent variables to X_t, therefore inherits the Markov property of X_t. On the interior of the state space of X_t, the ratio between f and h is positive, so, for example, a strictly positive f implies a strictly positive h.

We can also assign probabilistic meaning to the transformed PDE given in Equation 4.10; this same equation (given sufficient regularity conditions) arises as the solution to the probabilistic problem:

$$h(\Delta, y) = E\left[g_h(W_{t+\Delta}) \exp\left(-\int_t^{t+\Delta} r_h(W_u)\, du\right) \mid W_t = y\right] \tag{4.12}$$

where W_t is a Brownian motion. Note, however, that although the independent variable in this equation is y, the process Y_t, defined by the change of independent variables applied to X_t, is in general not a Brownian motion, and may have a state space different than the state space of the Brownian motion (i.e., the real line). However, the change of variable techniques described so far show that the original pricing problem is equivalent to the problem of finding a functional of a Brownian motion, even when the state variable is not a Brownian motion.

In a term structure context, the pricing PDE for some models which may seem quite different at first can be transformed by change of variables to the same general PDE, with only the final condition differing. For example, in the scalar version of the linear-quadratic model of Ahn, Dittmar, and Gallant (2002), the

$r_h(y)$ coefficient is a quadratic function of y; the model of Vasicek (1977) transforms to the same PDE after application of the change of variables; in both cases, the final condition for a zero-coupon bond price is $g(x) = 1$, which implies that $g_h(y)$ is exponential quadratic. The pricing PDEs for the models of Cox, Ingersoll, and Ross (1985) and Ahn and Gao (1999) both transform to the case in which $r_h(y)$ contains a term proportional to y^2, a constant term, and a term proportional to $1/y^2$; the two models then differ (for pricing purposes) only in the specification of g_h. However, there also exist many other models that have not appeared in the literature, but that also transform to these cases.

As shown above, the scalar pricing PDE can be transformed to one in which only the $r_h(y)$ coefficient is specific to the particular diffusion and interest rate specification. It is also possible (and sometimes convenient) to change the general PDE to one in which this coefficient vanishes, but either the drift or diffusion coefficient is non-trivial. Consider a PDE already in the form specified by Equation 4.10, and the further change of dependent variable:

$$h(\Delta, y) = u(y)\, v(\Delta, y) \tag{4.13}$$

Expressed in terms of v, the pricing PDE is now:

$$u \frac{\partial v}{\partial \Delta}(\Delta, y) = \left[\frac{\partial u}{\partial y}\right] \frac{\partial v}{\partial y}(\Delta, y) + \frac{u}{2}\frac{\partial^2 v}{\partial y^2}(\Delta, y) - \left[r_h(y)\, u - \frac{1}{2}\frac{\partial^2 u}{\partial y^2}\right] v(\Delta, y)$$

The coefficient on v in the transformed equation is zero if u is chosen carefully; the PDE can then be changed by either of the two changes of independent variable discussed above, to one in which the coefficient on v is zero with either the first right-hand term equal to zero or the second normalized to 1/2.

All of the changes of dependent and independent variable specified above are time-homogeneous. A transformation of independent variable which is not time-homogenous, and a corresponding transformation of the time variable, are often useful as well. Given a PDE of the form:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - r(y)\, h(\Delta, y) \tag{4.14}$$

we can transform the PDE in a way which effectively changes the $r(y)$ coefficient by any arbitrary linear function of y with the following change of variables:

$$z(\Delta, y) = y - \frac{b}{2}\Delta^2 \tag{4.15}$$

$$h(\Delta, y) = e^{\frac{b^2}{6}\Delta^3 - (a+by)\Delta} w(\Delta, z) \tag{4.16}$$

The transformed PDE is now:

$$\frac{\partial w}{\partial \Delta}(\Delta, z) = \frac{1}{2}\frac{\partial^2 w}{\partial z^2}(\Delta, z) - [r(y) - (a + by)]\, w(\Delta, z)$$

If $r(y)$ is affine in y (including the special case in which $r(y)$ is a constant), this transformation can reduce Equation 4.14 to the standard heat equation. This transformation can be used when $r(y)$ is not affine in y, but in this case, the coefficient on w is not a function of z only, but also depends on Δ.

Changing the r coefficient by an arbitrary quadratic function of y is also possible, through the following changes of variable, $b \neq 0$:

$$\begin{aligned} z(\Delta, y) &= \sqrt{-2b}\phi + \sqrt{-2b}(y-\phi)e^{b\Delta} && (4.17)\\ \tau(\Delta) &= 1 - e^{2b\Delta} && (4.18)\\ h(\Delta, y) &= e^{\frac{b}{2}\left[(y-\phi)^2 + d\Delta\right]} w(\tau, z) && (4.19)\end{aligned}$$

Note that the change of time variable is simply the basic time transformation of Section 3, with $k = -2b$. A common technique used in Section 5 is to use this change of variables, and, apply Theorem 2 to the resulting function to establish uniform convergence of a power series approximation in τ on $\Delta \in [0, +\infty)$. The resulting PDE can be expressed as:

$$\frac{\partial w}{\partial \tau}(\tau, z) = \frac{1}{2}\frac{\partial^2 w}{\partial z^2}(\tau, z) + \frac{e^{-2b\Delta}}{2b}\left[r(y) + \frac{b^2}{2}(y-\phi)^2 + \frac{b}{2}(1-d)\right] w(\tau, z) \tag{4.20}$$

This technique can be used to eliminate constant, linear, and quadratic terms from $r(y)$; however, if there are other terms, the transformed coefficient may no longer be time-homogeneous. We also note that, if b is real, the function added to the coefficient on w as a result of the transformation is always a quadratic function of y with a positive coefficient on the quadratic term. A quadratic function with a negative quadratic term can be added by choosing b to be an imaginary number; the resulting PDE remains perfectly valid as a mathematical problem, but a probabilistic interpretation is not obvious in this case.

An exception to the general non-time-homogeneity of Equation 4.20 is the case in which the only other terms contained in $r(y)$ are inversely proportional to the square of $(y-\phi)$:

$$r(y) = \frac{b^2}{2}(y-\phi)^2 + \left(\frac{b}{2} - d\right) + \frac{a}{(y-\phi)^2} \tag{4.21}$$

In this case, Equation 4.20 then becomes:

$$\frac{\partial w}{\partial \tau}(\tau, z) = \frac{1}{2}\frac{\partial^2 w}{\partial z^2}(\tau, z) - e^{-2b\Delta}\frac{a}{(y-\phi)^2} w(\tau, z) = \frac{1}{2}\frac{\partial^2 w}{\partial z^2}(\tau, z) - \frac{a}{(z-\phi)^2} w(\tau, z) \tag{4.22}$$

This particular case arises in the study of scalar affine process, but also in many other cases as well.

4.2 Existence of Analytic Solutions

It may not be obvious, for some choices of the $\mu(x)$, $\sigma(x)$, and $r(x)$ functions, that there are any final conditions $g(x)$ at all for which the corresponding solution $f(\Delta, x)$ of Equation 4.54 is analytic in the time variable in some neighborhood about the origin. Analysis of this issue is easiest if we assume that the pricing PDE has been transformed (by change of variables discussed in Section 4.1) into the form:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{\sigma^2(y)}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) \tag{4.23}$$

Continuity of the $\sigma(y)$ function, and positivity on the interior of the state space suffice for the existence of an infinite-dimensional set of final conditions, such that the solution to Equation 4.23 (with final condition) is not only analytic in a neighborhood of the origin, but entire. This can be seen by a simple construction. We begin with:

$$a_0(y) = 1 \tag{4.24}$$

$$b_0(y) = y \tag{4.25}$$

Given $a_n(y)$ and $b_n(y)$ defined for some integer $n \geq 0$, we define:

$$a_{n+1}(y) = \int_{y_0}^{y}\int_{y_0}^{v} \frac{a_n(u)}{\sigma^2(u)} du dv \tag{4.26}$$

$$b_{n+1}(y) = \int_{y_0}^{y}\int_{y_0}^{v} \frac{b_n(u)}{\sigma^2(u)} du dv \tag{4.27}$$

where y_0 is chosen arbitrarily. We now note that the functions:

$$h_{a,n}(\Delta, y) = \sum_{i=0}^{n} \frac{a_i(y)}{(n-i)!} \left(\frac{\Delta}{2}\right)^{(n-i)} \tag{4.28}$$

$$h_{b,n}(\Delta, y) = \sum_{i=0}^{n} \frac{b_i(y)}{(n-i)!} \left(\frac{\Delta}{2}\right)^{(n-i)} \tag{4.29}$$

are solutions to Equation 4.23 which satisfy the final conditions:

$$h_{a,n}(0, y) = a_n(y) \tag{4.30}$$

$$h_{b,n}(0, y) = b_n(y) \tag{4.31}$$

The $h_{a,n}(\Delta, y)$ and $h_{b,n}(\Delta, y)$ are polynomials in Δ, and are therefore entire functions. Also note that finite linear combinations of the $h_{a,n}(\Delta, y)$ and $h_{b,n}(\Delta, y)$:

$$h(\Delta, y) = \sum_{j=0}^{k} c_i h_{a,j}(\Delta, y) + \sum_{j=0}^{k} d_i h_{b,j}(\Delta, y) \tag{4.32}$$

are also solutions to Equation 4.23 which satisfy the final condition:

$$h(0, y) = \sum_{j=0}^{k} c_i a_j(y) + \sum_{j=0}^{k} d_i b_j(y) \tag{4.33}$$

Such functions are also entire in Δ. It is clear that $a_n(y)$ and $b_n(y)$ form an infinite-dimensional space of functions. Suppose all $a_n(y)$ and $b_n(y)$ are linearly independent for $n \leq m$. This condition is trivially satisfied for $m = 0$. Then if there were some c_i and d_i, $0 \leq i \leq m$, such that $a_{m+1}(y)$ could be expressed as:

$$a_{m+1}(y) = \sum_{j=0}^{m} c_i a_n(y) + \sum_{j=0}^{m} d_i b_n(y) \tag{4.34}$$

Then:

$$h(\Delta, y) = h_{a,m+1}(\Delta, y) - \sum_{j=0}^{m} c_i h_{a,n}(\Delta, y) - \sum_{j=0}^{m} d_i h_{b,n}(\Delta, y) \tag{4.35}$$

is a solution to Equation 4.23 with $h(0, y) = 0$. But the only analytic solution to the PDE which satisfies this final condition is $h(\Delta, y) = 0$. Since this is clearly not the case (the first term contains a sub-term with Δ to the power of $m+1$, and none of the other terms do), $h_{a,m+1}(\Delta, y)$ is linearly independent of the $a_n(y)$ and $b_n(y)$, $n \leq m$. The function $h_{b,m+1}(\Delta, y)$ can be shown to be linearly independent by the same method. By induction, we find that, no matter how large a value of m we choose, there are always final conditions which correspond to analytic solutions which are linearly independent of the other solutions.

Infinite linear combinations $h_{a,n}(\Delta, y)$ of the $h_{b,n}(\Delta, y)$, provided they converge, are also solutions to Equation 4.23; it is entirely possible that an infinite linear combination may converge for some combinations of values of Δ and y but not for others.

Although we have now characterized those final conditions for which the solution to Equation 4.23 is analytic in Δ, this characterization may not always be useful in practice. It is relatively straightforward to construct final conditions that generate analytic solutions, but it may not be obvious how to take a given final condition and determine whether it is in fact spanned by the a_n and b_n functions specified above. However, in several special cases, there are other techniques for characterizing the set of final conditions which correspond to analytic (in Δ) solutions. The following sections explore several such cases.

4.3 Brownian Motion

The special case of Equation 4.14 in which the $r(y)$ coefficient is equal to zero:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) \tag{4.36}$$

admits particularly simple analysis. The region of analyticity for a solution $h(\Delta, y)$ of Equation 4.36 can be characterized in a straightforward manner, as shown in the following theorem.

Theorem 3. *Let $g(y)$ be an entire function, let $c > 0$ be a positive constant, and let $k(\alpha)$ be a positive real continuous function defined on $\alpha \in [0, \pi]$ with $k(0) = k(\pi)$, such that:*

$$\left|g\left(se^{i\alpha}\right)\right| \leq ce^{\frac{s^2}{2k(\alpha)}} \tag{4.37}$$

for all real s and $\alpha \in [0, \pi]$. Then there exists a function $h(\Delta, y)$ such that:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) \tag{4.38}$$

$$h(0, y) = g(y) \tag{4.39}$$

that is defined and analytic for all complex Δ and y such that $\Delta = re^{i\theta}$ with $\theta \in [0, 2\pi]$ and $0 \leq r < k\left(\frac{\theta}{2}\right)$.

Proof: See appendix.

In probabilistic terms, this theorem describes a large class of functions of a Brownian motion whose conditional expectations are analytic in the time variable, and characterizes the region of analyticity. However, it also applies to many other situations. For example, a process which is not a Brownian motion, but that can be changed to a Brownian motion by change of independent variable, is also covered by applying Theorem 3 after the change of variables. Similarly, this theorem effectively characterizes a set of final asset payoffs which generate pricing functions which are analytic in maturity, provided the pricing PDE can be converted to Equation 4.36 by change of dependent and/or independent variables, as in Colton (1979). For any of these applications, if it can be established through the theorem that the region of analyticity includes a neighborhood of the positive real axis $[0, T]$ for some $0 < T < +\infty$, then series approximations that converge uniformly on this interval can be constructed by the non-affine transformation of the time variable as described in Section 3.

It may be useful to characterize those final conditions that correspond to solutions $h(\Delta, y)$ of Equation 4.36 that are entire in Δ. The following corollary examines this case:

Corollary 2. *Let $g(y)$ be an entire function, and for each positive real $k > 0$, let there be a positive real constant $c_k > 0$ such that:*

$$\left|g\left(se^{i\alpha}\right)\right| \leq c_k e^{\frac{s^2}{2k}} \tag{4.40}$$

for all real s and all $\alpha \in [0, \pi]$. Then there exists a function $h(\Delta, y)$ such that:

$$\begin{aligned} \frac{\partial h}{\partial \Delta}(\Delta, y) &= \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) && (4.41) \\ h(0, y) &= g(y) && (4.42) \end{aligned}$$

that is defined and analytic for all complex Δ and y.

Proof: See appendix.

This corollary applies to all the same situations described in the discussion of Theorem 3, provided a stronger growth restriction on the final condition is imposed. But, if the conditions of the corollary are satisfied, then the conditional moment or pricing function is analytic for all complex values of the time variable, and a series approximation to the desired function converges uniformly on any portion of the positive real axis $[0, T]$, for any particular value $0 < T < +\infty$.

The applicability of the previous two results can be greatly extended if they are applied after one of the time-inhomogeneous time transformations discussed in Section 4.1; as we will see in Section 5, several term structure models that have appeared in the literature are then covered (as are many models that have not previously appeared in the literature). As discussed in Section 4.1, it is possible to eliminate the last term in the PDE:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - r(y)\, h(\Delta, y) \tag{4.43}$$

by change of variables (that depend explicitly on time), if $r(y)$ is a quadratic function of y. After the change of variables, Theorem 3 and Corollary 2 can be applied, but with τ from Equation **??** in place of Δ. The following two results express the region of analyticity directly in terms of Δ. The first result considers the special case in which the function $r(y)$ is affine in y; the region of analyticity is then the same as if $r(y)$ were equal to zero, as per the following corollary:

Corollary 3. *Let $g(y)$ be an entire function, let $c > 0$ be a positive constant, and let $k(\alpha)$ be a positive real continuous function defined on $\alpha \in [0, \pi]$ with $k(0) = k(\pi)$, such that:*

$$\left|g\left(se^{i\alpha}\right)\right| \leq ce^{\frac{r^2}{2k(\alpha)}} \tag{4.44}$$

for all real s and $\alpha \in [0, \pi]$. Then there exists a function $h(\Delta, y)$ such that:

$$\begin{aligned} \frac{\partial h}{\partial \Delta}(\Delta, y) &= \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) + (\phi y + \varphi) h(\Delta, y) && (4.45) \\ h(0, y) &= g(y) && (4.46) \end{aligned}$$

that is defined and analytic for all complex Δ and y such that $\Delta = re^{i\theta}$ with $\theta \in [0, 2\pi]$ and $0 \leq r < k\left(\frac{\theta}{2}\right)$.

Proof: See appendix.

Although we do not state and prove so formally, a result analogous to Corollary 2 can also be shown for the case in which $r(y)$ is an affine function of y; the region of analyticity is then unaffected by the presence of the non-zero $r(y)$ term in the PDE.

When $r(y)$ includes a non-zero quadratic term, determining the region of analyticity is considerably more involved. The following result expresses the region of analyticity directly in terms of Δ.

Corollary 4. *Let $g(y)$ be an entire function, let $c > 0$ be a positive constant, let $\kappa \neq 0$ be any non-zero complex number, and let $k(\alpha)$ be a positive real continuous function defined on $\alpha \in [0, \pi]$, such that:*

$$\left|e^{-\frac{\kappa}{2}s^2 e^{2i\alpha}} g\left(se^{i\alpha}\right)\right| \leq ce^{\frac{s^2}{2k(\alpha)}} \tag{4.47}$$

for all real s and $\alpha \in [0, \pi]$. Then there exists a function $w(\tau, z)$ such that:

$$\begin{aligned} \frac{\partial w}{\partial \tau}(\tau, z) &= \frac{1}{2}\frac{\partial^2 w}{\partial z^2}(\tau, z) && (4.48) \\ w(0, z) &= e^{-\frac{\kappa}{2}(z-\phi)^2} g(z) && (4.49) \end{aligned}$$

that is analytic for all complex τ and z such that $\tau = re^{i\theta}$ with $\theta \in [0, 2\pi]$ and $0 \leq r < k\left(\frac{\theta}{2}\right)$. Furthermore, the function $h(\Delta, y)$ defined by:

$$h(\Delta, y) = e^{\frac{\kappa}{2}(y-\phi)^2 - \left(\varphi - \frac{\kappa}{2}\right)\Delta} w\left(\frac{e^{2\kappa\Delta} - 1}{2\kappa}, \phi + e^{\kappa\Delta}(y - \phi)\right) \tag{4.50}$$

satisfies:

$$\begin{aligned} \frac{\partial h}{\partial \Delta}(\Delta, y) &= \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left[\frac{\kappa^2}{2}(y - \phi)^2 + \varphi\right] h(\Delta, y) && (4.51) \\ h(0, y) &= g(y) && (4.52) \end{aligned}$$

for all complex Δ *and* y *such that:*

$$\frac{e^{2\kappa\Delta}-1}{2\kappa}=re^{i\theta} \tag{4.53}$$

for some r *and* θ*, with* $\theta\in[0,2\pi]$ *and* $0\leq r<k\left(\frac{\theta}{2}\right)$*.*

Proof: See appendix.

We note that Corollary 4 can be applied to a PDE of the form indicated in Equation 4.51 with b replaced by $-b$, since this leaves the PDE unaltered.

Note the symmetry in the regions of analyticity as described in Theorem 3 and Corollaries 2 and 3; analyticity along a straight line form the origin to some value $\Delta=re^{i\theta}$ requires a boundedness condition on the growth of the final condition $g(y)$ for $y=se^{i\theta/2}$ as goes to $\pm\infty$, and the growth condition is the same regardless of θ. However, although the growth restrictions required for Corollary 4 are also symmetric in θ, the angle in the complex plane denoted by θ is not the angle of Δ, but of a non-affine function of Δ. The growth restrictions are therefore not radially symmetric in the angle of Δ. For example, suppose b is real and negative. For real and positive values of Δ, the value of r varies from 0 (at $\Delta=0$) to $-1/(2b)$ (at $\Delta=+\infty$). Therefore, provided $k(0)$ is sufficiently large, the region of analyticity extends to $\Delta=+\infty$ along the positive real axis. This result stands in contrast to Theorem 3 and Corollary 3, for which there is no finite value of $k(0)$ that guarantees analyticity out to $\Delta=+\infty$. Considering negative real values of Δ instead, we note an asymmetry; there is no value of $k(\pi/2)$ that guarantees convergence out to $\Delta=-\infty$. Many problems involving conditional moments or bond pricing with a mean-reverting process are covered by Corollary 4 (after a change of variables), although other cases are covered as well. In Section 5, we find that this corollary applies to the bond prices under the term structure models of Vasicek (1977) and Ahn, Dittmar, and Gallant (2002).

4.4 General Affine

The following partial differential equation may be considered a generalization of the Brownian motion case considered in the previous section:

$$\frac{\partial h}{\partial\Delta}(\Delta,y)=\frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta,y)+\frac{a}{y^2}h(\Delta,y) \tag{4.54}$$

This case clearly nests the Brownian motion case of the preceding section, since we can set $a=0$. We refer to this equation as the general affine PDE; although there may at first or even second glance seem to be nothing particularly affine about Equation 4.54, the problem of finding conditional moments for any scalar affine diffusion can be reduced to solving this equation by the changes of variables discussed in Section 4.1. The pricing PDE for every scalar affine yield model can also be transformed to this PDE by change of variables, as can the pricing PDE for some non-affine models (e.g., Ahn and Gao (1999)). The final conditions which admit analytic (in Δ) solutions can be characterized explicitly in this case.

Theorem 4. *Let* $w_1(y)$ *and* $w_2(y)$ *be entire functions, let* $c>0$ *be a positive constant, and let* $k(\alpha)$ *be a* *positive real continuous function defined on $\alpha \in [0, \pi]$ with $k(0) = k(\pi)$, such that:*

$$\left|w_1\left(s^2 e^{2i\alpha}\right)\right| \leq ce^{\frac{s^2}{2k(\alpha)}} \tag{4.55}$$

$$\left|w_2\left(s^2 e^{2i\alpha}\right)\right| \leq ce^{\frac{s^2}{2k(\alpha)}} \tag{4.56}$$

for all real s and $\alpha \in [0, \pi]$. Then there exists a solution to the problem:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) + \frac{a}{y^2} h(\Delta, y) \tag{4.57}$$

$$h(0, y) = g(y) \tag{4.58}$$

where:

$$g(y) = w_1\left(y^2\right) y^{\frac{1-\sqrt{1-8a}}{2}} + w_2\left(y^2\right) y^{\frac{1+\sqrt{1-8a}}{2}}, \quad \frac{\sqrt{1-8a}}{2} \notin \mathbb{N} \tag{4.59}$$

$$g(y) = w_1\left(y^2\right) y^{\frac{1-\sqrt{1-8a}}{2}} + w_2\left(y^2\right) y^{\frac{1+\sqrt{1-8a}}{2}} \ln y, \quad \frac{\sqrt{1-8a}}{2} \in \mathbb{N} \tag{4.60}$$

that is defined and analytic for all complex Δ and $y \neq 0$ such that $\Delta = re^{i\theta}$ for some $\theta \in [0, 2\pi]$ and $0 \leq r < k\left(\frac{\theta}{2}\right)$. Note that a branch cut discontinuity in y is possible.

Proof: See appendix.

The above result, while valid for any value of a, may not be expressed in the most useful form for real $a > 1/8$, since the two terms in the expression for $h(0, y)$ are then, in general, complex, even if w_1 and w_2 are real functions; for most applications, we are interested in real-valued final conditions. However, the final condition can be restated equivalently as:

$$h(0, y) = w_3\left(y^2\right) y^{\frac{1}{2}} \cos\left(\frac{\sqrt{8a-1}}{2} \ln y\right) + w_4\left(y^2\right) y^{\frac{1}{2}} \sin\left(\frac{\sqrt{8a-1}}{2} \ln y\right) \tag{4.61}$$

where

$$w_3(z) = \frac{w_1(z) + w_2(z)}{\sqrt{2}} \tag{4.62}$$

$$w_4(z) = \frac{\imath w_2(z) - \imath w_1(z)}{\sqrt{2}} \tag{4.63}$$

or, equivalently:

$$w_1(z) = \frac{w_3(z) + \imath w_4(z)}{\sqrt{2}} \tag{4.64}$$

$$w_2(z) = \frac{w_3(z) - \imath w_4(z)}{\sqrt{2}} \tag{4.65}$$

If the functions w_1 and w_2 satisfy the growth conditions of Equations 4.55 and 4.56, then w_3 and w_4 satisfy the same growth conditions (possibly with a different value of c); the final condition $g(y)$ is then real, provided $a > 1/8$ and w_3 and w_4 are real functions.

As in the Brownian motion case, we may interpret solutions to Equation 4.54 as functionals of a Brownian motion, but in this case they are not ordinary conditional moments, but discounted future payoffs:

$$h(\Delta, y) = E\left[g(W_{t+\Delta})\exp\left(\int_t^{t+\Delta}\frac{a}{W_u^2}du\right) \mid W_t = y\right] \tag{4.66}$$

As in the Brownian motion case, the theorem describes a large class of final conditions such that the corresponding solutions $h(\Delta, y)$ are analytic in the time variable, and characterizes the region of analyticity. Also as in the Brownian motion case, it also applies to many other situations. For example, conditional moments of the square-root process of Feller (1951) (after changes of both independent and dependent variables) satisfy this PDE. Furthermore, conditional moments of any process which can be changed to the square-root process by a change of independent variable are also covered the theorem. Similarly, the theorem effectively characterizes a set of final asset payoffs which generate pricing functions which are analytic in maturity, for a wide combination of process and interest rate specifications, provided the pricing PDE can be converted to Equation 4.54 by change of dependent and/or independent variables, as in Colton (1979). For any of these applications, if it can be established through the theorem that the region of analyticity includes a neighborhood of the positive real axis $[0, T]$ for some $0 < T < +\infty$, then series approximations that converge uniformly on this interval can be constructed by the non-affine transformation of the time variable as described in Section 3.

As in the Brownian motion case, it may be useful to characterize those final conditions that correspond to solutions $h(\Delta, y)$ to Equation 4.54 that are entire in Δ. The following corollary examines this case:

Corollary 5. *Let $w_1(y)$ and $w_2(y)$ be entire functions, and for each positive real $k > 0$, let there be a positive real constant $c_k > 0$ such that:*

$$\left|w_1\left(s^2 e^{2i\alpha}\right)\right| \leq c_k e^{\frac{s^2}{2k}} \tag{4.67}$$

$$\left|w_2\left(s^2 e^{2i\alpha}\right)\right| \leq c_k e^{\frac{s^2}{2k}} \tag{4.68}$$

for all real s and all $\alpha \in [0, \pi]$. Then there exists a function $h(\Delta, y)$ such that:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) + \frac{a}{y^2}h(\Delta, y) \tag{4.69}$$

$$h(0, y) = g(y) \tag{4.70}$$

where:

$$g(y) = w_1\left(y^2\right)y^{\frac{1-\sqrt{1-8a}}{2}} + w_2\left(y^2\right)y^{\frac{1+\sqrt{1-8a}}{2}}, \quad \frac{\sqrt{1-8a}}{2} \notin \mathbb{N} \tag{4.71}$$

$$g(y) = w_1\left(y^2\right)y^{\frac{1-\sqrt{1-8a}}{2}} + w_2\left(y^2\right)y^{\frac{1+\sqrt{1-8a}}{2}}\ln y, \quad \frac{\sqrt{1-8a}}{2} \in \mathbb{N} \tag{4.72}$$

that is defined and analytic for all complex Δ and $y \neq 0$. Note that a branch cut discontinuity in y is possible.

Proof: See appendix.

This corollary applies to all the same situations described in the discussion of Theorem 4, provided a stronger growth restriction on the final condition is imposed. But, if the conditions of the corollary apply, then the conditional moment or pricing function is analytic for all complex values of the time variable, and a series approximation to the desired function converges uniformly on any portion of the positive real axis $[0, T]$, for any particular value $0 < T < +\infty$.

As in the Brownian motion case, the previous two results can be applied more broadly if a time-inhomogeneous time transformations from Section 4.1 is applied to the PDE first. As previously noted (and shown in detail in Section 5), several term structure models that have appeared in the literature reduce to the Brownian motion case after changes of variables; several more reduce to the general affine case (as do many other models that have not previously appeared in the literature). As discussed in Section 4.1, it is possible to eliminate two of the three terms in the $r(y)$ coefficient from the PDE:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) + \left(\frac{a}{y^2} - \frac{b^2}{2}y^2 + d\right) h(\Delta, y) \tag{4.73}$$

by change of variables (that depend explicitly on time); Theorem 4 and Corollary 5 can then be applied with τ in place of Δ. The following two results express the region of analyticity directly in terms of Δ. The first result considers the special case in which the $b = 0$; the presence of the d parameter then has no effect on the region of analyticity, as per the following corollary:

Corollary 6. *Let $w_1(y)$ and $w_2(y)$ be entire functions, let $c > 0$ be a positive constant, and let $k(\alpha)$ be a positive real continuous function defined on $\alpha \in [0, \pi]$ with $k(0) = k(\pi)$, such that:*

$$\begin{aligned}\left|w_1\left(s^2 e^{2i\alpha}\right)\right| &\leq ce^{\frac{s^2}{2k(\alpha)}} &(4.74)\\ \left|w_2\left(s^2 e^{2i\alpha}\right)\right| &\leq ce^{\frac{s^2}{2k(\alpha)}} &(4.75)\end{aligned}$$

for all real s and $\alpha \in [0, \pi]$. Then there exists a function $h(\Delta, y)$ such that:

$$\begin{aligned}\frac{\partial h}{\partial \Delta}(\Delta, y) &= \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) + \left(\frac{a}{y^2} + d\right) h(\Delta, y) &(4.76)\\ h(0, y) &= g(y) &(4.77)\end{aligned}$$

where:

$$\begin{aligned}g(y) &= w_1\left(y^2\right) y^{\frac{1-\sqrt{1-8a}}{2}} + w_2\left(y^2\right) y^{\frac{1+\sqrt{1-8a}}{2}}, \quad \frac{\sqrt{1-8a}}{2} \notin \mathbb{N} &(4.78)\\ g(y) &= w_1\left(y^2\right) y^{\frac{1-\sqrt{1-8a}}{2}} + w_2\left(y^2\right) y^{\frac{1+\sqrt{1-8a}}{2}} \ln y, \quad \frac{\sqrt{1-8a}}{2} \in \mathbb{N} &(4.79)\end{aligned}$$

that is defined and analytic for all complex Δ and $y \neq 0$ such that $\Delta = re^{i\theta}$ for some $\theta \in [0, 2\pi]$ and $0 \leq r < k\left(\frac{\theta}{2}\right)$. Note that a branch cut discontinuity in y is possible.

Proof: See Appendix.

It is, as in the previous section, possible to extend this result analogously to Corollary 5 when the $r(y)$ function contains a constant term; the region of analyticity is then unaffected by the presence of the additional term in the PDE.

When $r(y)$ includes a non-zero quadratic term, the general affine case behaves similarly to the Brownian motion case; the symmetry of Theorem 4 and Corollaries 5 and 6 is then lost. The following result expresses the region of analyticity directly in terms of Δ.

Corollary 7. *Let $w_1(y)$ and $w_2(y)$ be entire functions, let $c > 0$ be a positive constant, let $b \neq 0$ be any non-zero complex number, and let $k(\alpha)$ be a positive real continuous function defined on $\alpha \in [0, \pi]$, such that:*

$$\left| e^{-\frac{b}{2}s^2 e^{2i\alpha}} w_1\left(s^2 e^{2i\theta}\right) \right| \leq c e^{\frac{s^2}{2k(\alpha)}} \tag{4.80}$$

$$\left| e^{-\frac{b}{2}s^2 e^{2i\alpha}} w_2\left(s^2 e^{2i\theta}\right) \right| \leq c e^{\frac{s^2}{2k(\alpha)}} \tag{4.81}$$

for all real s and $\alpha \in [0, \pi]$. Then there exists a function $w(\tau, z)$ such that:

$$\frac{\partial w}{\partial \tau}(\tau, z) = \frac{1}{2}\frac{\partial^2 w}{\partial z^2}(\tau, z) + \frac{a}{z^2} w(\tau, z) \tag{4.82}$$

$$w(0, z) = e^{-\frac{\kappa}{2}(z-\theta)^2} g(z) \tag{4.83}$$

where:

$$g(z) = w_1\left(z^2\right) z^{\frac{1-\sqrt{1-8a}}{2}} + w_2\left(z^2\right) z^{\frac{1+\sqrt{1-8a}}{2}}, \quad \frac{\sqrt{1-8a}}{2} \notin \mathbb{N} \tag{4.84}$$

$$g(y) = w_1\left(z^2\right) z^{\frac{1-\sqrt{1-8a}}{2}} + w_2\left(z^2\right) z^{\frac{1+\sqrt{1-8a}}{2}} \ln z, \quad \frac{\sqrt{1-8a}}{2} \in \mathbb{N} \tag{4.85}$$

that is analytic for all complex τ and $z \neq 0$ such that $\tau = re^{i\theta}$ with $\theta \in [0, 2\pi]$ and $0 \leq r < k\left(\frac{\theta}{2}\right)$. Furthermore, the function $h(\Delta, y)$ defined by:

$$h(\Delta, y) = e^{\frac{b}{2}y^2 + \left(\frac{b}{2} - d\right)\Delta} w\left(\frac{e^{2b\Delta} - 1}{2b}, e^{b\Delta} y\right) \tag{4.86}$$

satisfies:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left(\frac{a}{y^2} + \frac{b^2}{2}y^2 + d\right) h(\Delta, y) \tag{4.87}$$

$$h(0, y) = g(y) \tag{4.88}$$

for all complex Δ and y such that:

$$\frac{e^{2b\Delta} - 1}{2b} = re^{i\theta} \tag{4.89}$$

for some r and θ, with $\theta \in [0, 2\pi]$ and $0 \leq r < k\left(\frac{\theta}{2}\right)$. Note that a branch cut discontinuity in y is possible.

Proof: See appendix.

As with Corollary 4, the growth restrictions imposed by 7 are not symmetric in the angle of Δ; if b is real and negative, for sufficiently large $k(0)$, the region of analyticity extends to $\Delta = +\infty$ along the positive real axis; in, for example, the negative real direction, there is no value of $k(\pi/2)$ sufficiently large to guarantee convergence to $\Delta = -\infty$. Many problems involving conditional moments or bond pricing with a mean-reverting process are covered by Corollary 7 (after a change of variables), although other cases are covered as well. In Section 5, we find that this corollary applies to the bond prices under the term structure models of Cox, Ingersoll, and Ross (1985) and Ahn and Gao (1999).

5 Examples

In this section, we consider several specifications of the pricing PDE (or the special case of the Chapman-Kolmogorov backward equation), and examine the behavior of power series solutions using the basic time transformation, within a small (i.e., radius $r < 1$) circle, within a large circle (i.e., radius $r > 1$), or when used twice to form a compound time transform.

5.1 Vasicek

The interest rate and bond pricing model of Vasicek (1977), the instantaneous interest rate follows a (non-centered) Ornstein-Uhlenbeck process:

$$dX_t = (a + bX_t)\, dt + \sigma dW_t \tag{5.1}$$

In this case, the interest rate is equal to the state variable itself:

$$r(x) = x \tag{5.2}$$

The partial differential equation satisfied by bond (and other asset prices) is therefore:

$$\begin{aligned} \frac{\partial f}{\partial \Delta}(\Delta, x) &= (a + bx)\frac{\partial f}{\partial x}(\Delta, x) + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}(\Delta, x) - xf(\Delta, x) && (5.3)\\ f(0, x) &= g(x) && (5.4) \end{aligned}$$

where $g(x)$ is the final asset payoff, for example, $g(x) = 1$ for zero-coupon bonds. Conditional moments satisfy the same partial differential equation with the last term deleted. Centered conditional moments can be found in closed-form; the first and second (centered) conditional moments are given by:

$$\begin{aligned} E_t[X_{t+\Delta}] &= \left(X_t + \frac{a}{b}\right)\exp(b\Delta) - \frac{a}{b} && (5.5)\\ E_t\left[X_{t+\Delta}^2\right] &= \left[\left(X_t + \frac{a}{b}\right)^2 + \frac{\sigma^2}{2b}\right]\exp(2b\Delta) - 2\frac{a}{b}\left(X_t + \frac{a}{b}\right)\exp(b\Delta) + \left(\frac{a^2}{b^2} - \frac{\sigma^2}{2b}\right) && (5.6) \end{aligned}$$

Zero-coupon bond prices are also known in closed-form:

$$P(\Delta, x) = \exp\left[\frac{1 - \exp(b\Delta)}{b}x - a\frac{\exp(b\Delta) - 1 - b\Delta}{b^2} + \frac{\sigma^2}{2b}\frac{\exp(2b\Delta) - 4\exp(b\Delta) + 3 + 2b\Delta}{2b^2}\right] \tag{5.7}$$

and by inspection, are not only analytic, but entire. To change the PDE into the form given by Equation 4.10, we make the changes of variables:

$$\begin{aligned} f(\Delta, x) &= e^{-\frac{b}{2\sigma^2}\left(x + \frac{a}{b}\right)^2} h(\Delta, y) && (5.8)\\ y &= \frac{x}{\sigma} && (5.9) \end{aligned}$$

The pricing PDE, expressed in terms of h and y instead of f and x, is then:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left[\frac{b^2}{2}\left(y + \frac{a}{b\sigma} + \frac{\sigma}{b^2}\right)^2 + \left(\frac{b}{2} - \frac{a}{b} - \frac{\sigma^2}{2b^2}\right)\right] h(\Delta, y) \tag{5.10}$$

$$h(0, y) = e^{\frac{b}{2}\left(y + \frac{a}{b\sigma}\right)^2} g(y\sigma) \tag{5.11}$$

For a zero-coupon bond, the final condition is:

$$h(0, y) = e^{\frac{b}{2}\left(y + \frac{a}{b\sigma}\right)^2} \tag{5.12}$$

This PDE and final condition together satisfy the conditions of Corollary 4 for $\kappa = b$, for any choice of $k(\alpha)$. Consequently, a power series expansion of $w(\tau, z)$ in τ converges uniformly within a circle of any radius. in this case. In the case $b < 0$, the interval $\Delta \in [0, +\infty)$ maps to the interval $\tau \in [0, -1/(2b))$; consequently, a power series approximation of $w(\tau, z)$ in τ converges uniformly on this interval. Since:

$$h(\Delta, y) = e^{\frac{b}{2}(y-\phi)^2 - \left(\varphi - \frac{b}{2}\right)\Delta} w(\tau, z)$$

the function $h(\Delta, y)$ (and therefore $f(\Delta, x)$ also) can be approximated uniformly on this interval provided $\frac{\sigma^2}{2b} > -a$.

We now examine numerically the accuracy of two methods of power series expansion: directly in Δ, and in τ instead. We choose parameter values $a = 0.04$, $b = -0.5$, and $\sigma = 0.03$, which are roughly realistic values for a US interest rate process. Taking the initial value $X_0 = 6\%$, the true expected value of the process is 6.7869% at $\Delta = 5$, 7.9865% at $\Delta = 10$, 7.9999% at $\Delta = 20$, and almost exactly 8% at $\Delta = 40$. A sixteen term power series expansion in Δ (i.e., including powers of Δ from 0 to 15) of the expected value matches the true values to the number of decimal places shown for $\Delta = 1$ and for $\Delta = 5$. For $\Delta = 10$, some small inaccuracy becomes evident; the value of the sixteen term expansion is 7.9978%. At $\Delta = 20$, the value of the sixteen term expansion is 604.988%, and at $\Delta = 40$ it is 28,208,100%, which is probably not accurate enough for most applications. For sixteen term expansions for the second moment, the divergence is more rapid; the true values at $\Delta = 1$, 5, 10, 20, and 40 are 0.0051752, 0.0070340, 0.0072784, 0.0072999, and 0.0073, respectively. The corresponding values of the expansion are 0.0051752, 0.0070368, 0.15654, 7070.99, and 2.97117×10^8. Figures 7 and 8 show these results graphically. As shown in Figure 7, the 16-term expansion for the mean performs well until the time horizon is eleven years or so; Figure 8 shows that the 16-term expansion for the second moment deteriorates after about seven years.

We do not show the accuracy of a power series expansion in τ, because such an expansion terminates after one term (i.e., after the constant term) in the first moment case, and after two terms (i.e., after the τ term) in the second moment case.

The more interesting case is that of bond prices. Figure 9 shows the accuracy of a power series expansion in Δ to a zero-coupon bond price. Although such an expansion converges for any value of Δ (since the bond price function is entire in Δ), even after 11 terms, the approximation begins to diverge noticably for $\Delta \geq 5$, and quite badly for $\Delta \geq 7$.

We do not show the accuracy of an expansion in τ graphically, because even with only one term, the

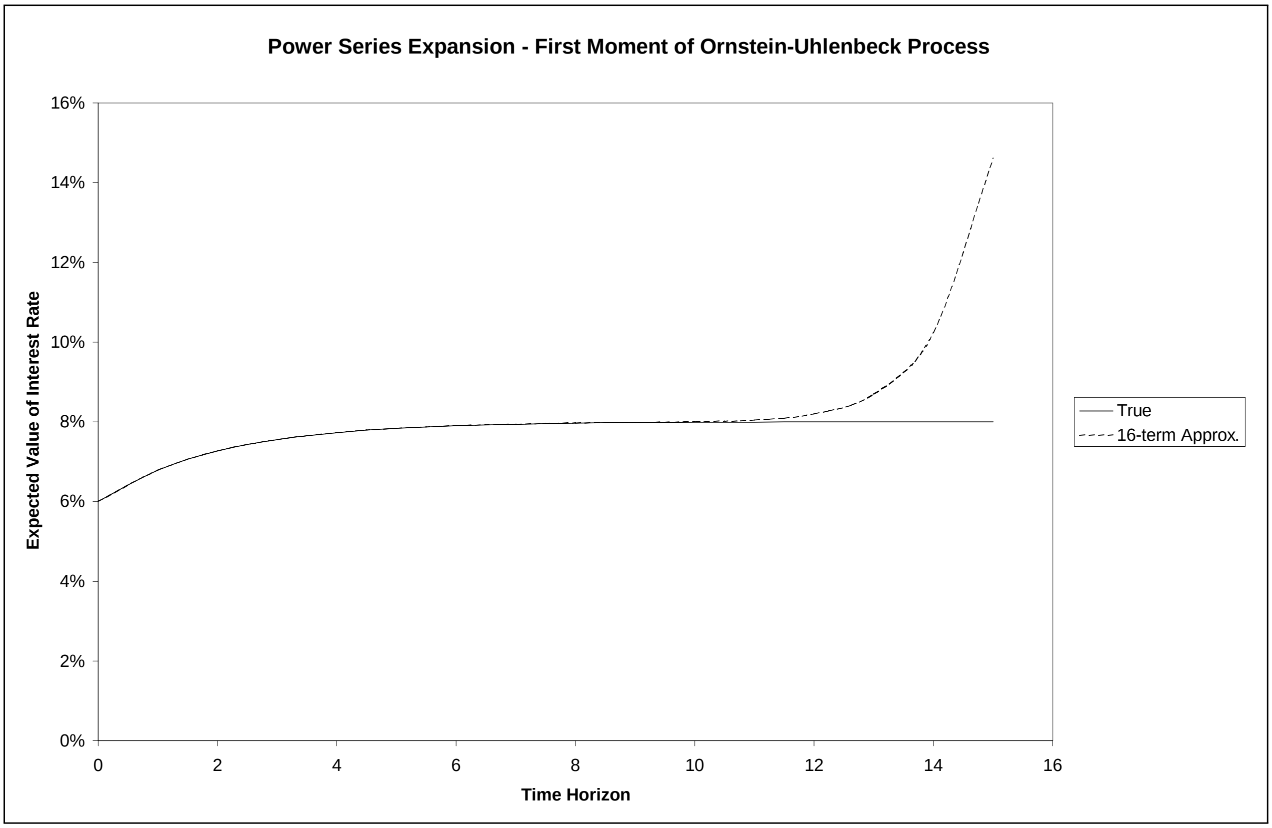

Figure 7: This figure shows the expected value of an Ornstein-Uhlenbeck process for varying time horizons, using the true conditional moment and a 16-term power series approximation. The initial value is 6%, the unconditional mean is 8%, the speed of mean reversion is 0.5, and the instantaneous standard deviation is 0.03. As shown, the 16-term approximation is very accurate up to a time horizon of about 10 years, but diverges badly from the true value for longer horizons.

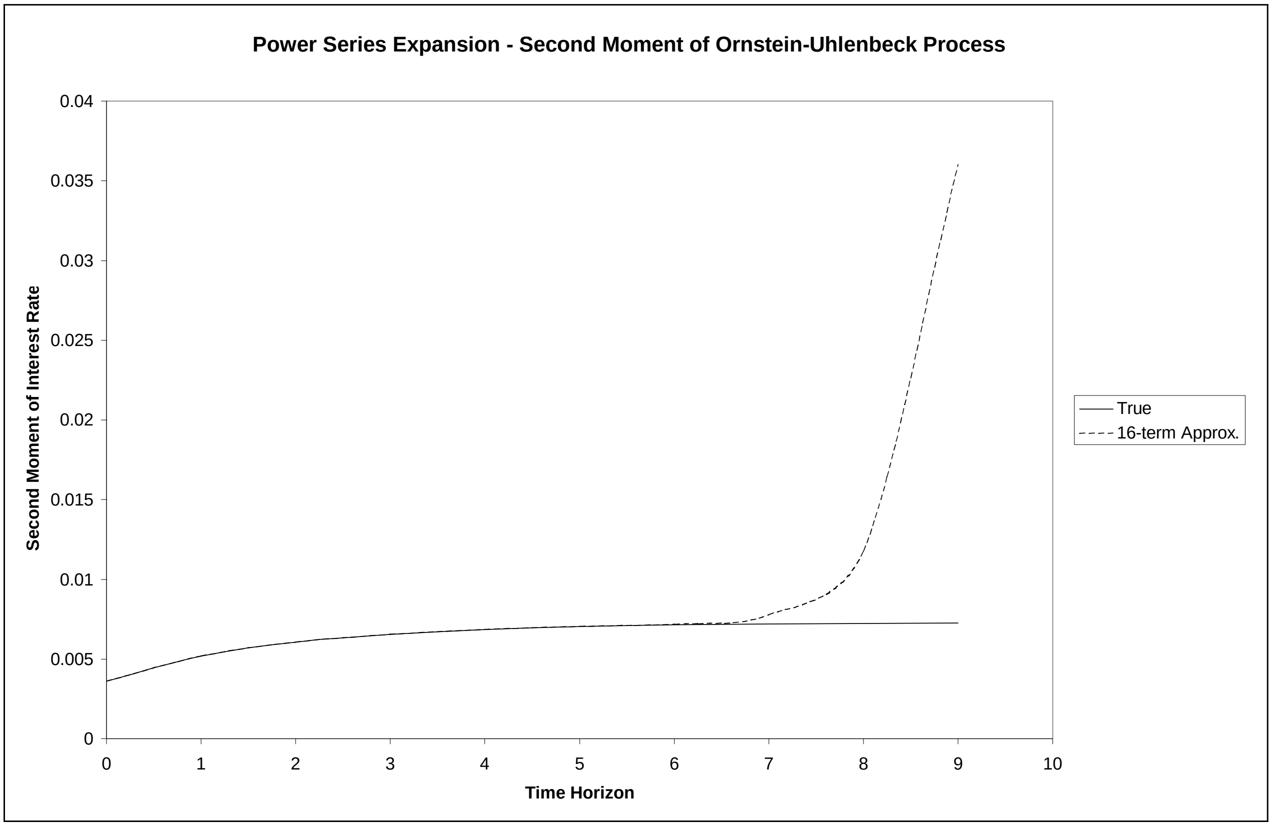

Figure 8: This figure shows the second moment of an Ornstein-Uhlenbeck process for varying time horizons, using the true conditional second moment and a 16-term power series approximation. The initial value of the interest rate is 6%, its unconditional mean is 8%, the speed of mean reversion is 0.5, and the instantaneous standard deviation is 0.03. As shown, the 16-term approximation is very accurate up to a time horizon of about 7 years, but diverges badly from the true value for longer horizons.

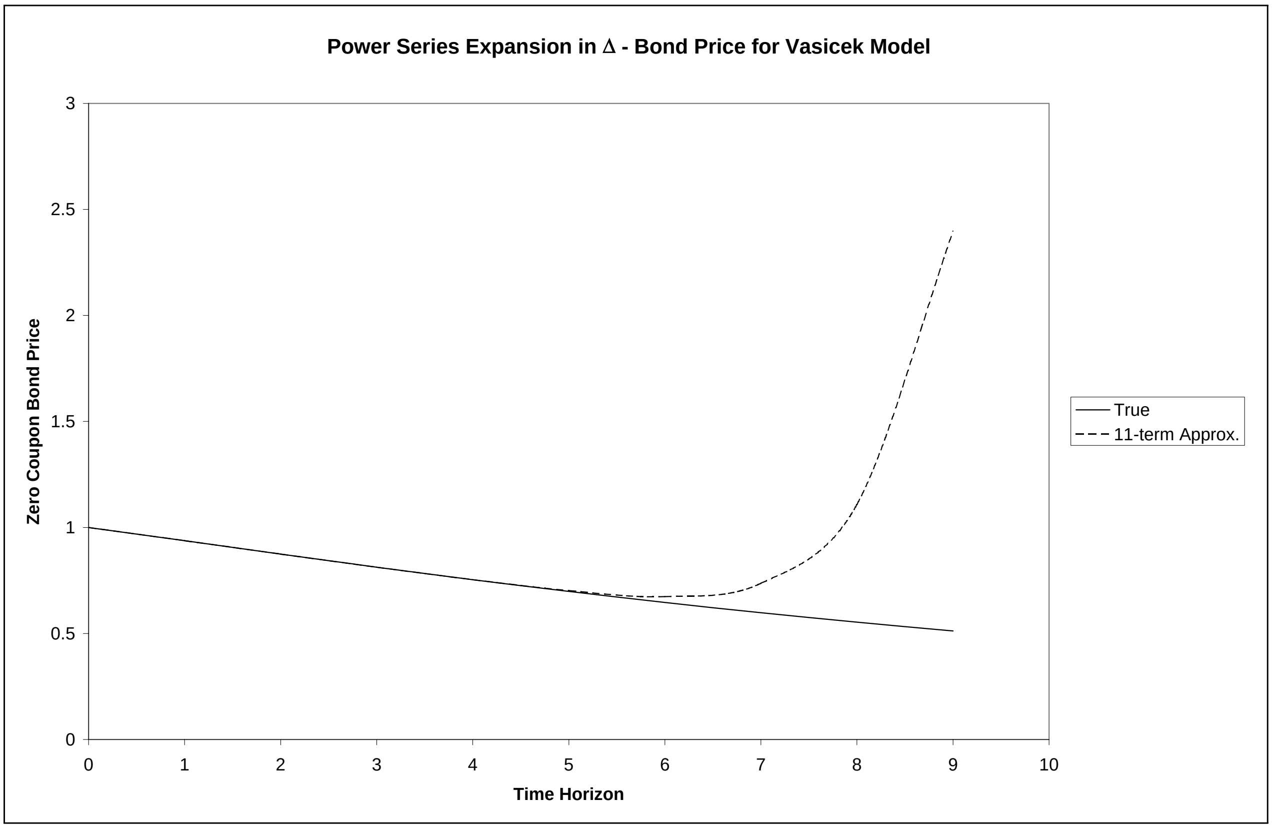

Figure 9: This figure shows the price of a zero-coupon bond in the Vasicek model, using the closed form expression and an 11-term power series approximation. The initial value of the interest rate is 6%, its unconditional mean is 8%, the speed of mean reversion is 0.5, and the instantaneous standard deviation is 0.03. As shown, the 11-term approximation is very accurate up to a time horizon of approximately 5 years, but diverges badly from the true value for longer horizons.

approximate price and true price are so close together the two lines would appear to coincide in a graph. The relative error (i.e., the difference between the approximate price and the true price, divided by the true price) is at most 0.0018, in the range $\Delta \in [0, 20]$. With a second term (i.e., the τ term), the largest relative error in this range is approximately 0.0000016.

Thus, in this case, in which closed form expressions for both conditional moments and bond prices are available, we find that a power series expansion approach in τ leads to remarkably close approximations even with a very small number of terms, for values of τ that correspond to large positive values of Δ. By contrast, power series expansions in Δ, although they do converge for any value of Δ, are quite inaccurate for large values of Δ, unless a very large (and impractical) number of terms are included in the approximation.

5.2 Linear-Quadratic

Ahn, Dittmar, and Gallant (2002) construct a term structure model in which the state variable, as in the model of Vasicek (1977), also follows an Ornstein-Uhlenbeck process:

$$dX_t = (a + bX_t)\, dt + \sigma dW_t \tag{5.13}$$

However, the instantaneous interest rate is a quadratic function in the state variable:[4]

$$r(x) = x^2 + d \tag{5.14}$$

The partial differential equation satisfied by bond (and other asset prices) is therefore:

$$\begin{aligned} \frac{\partial f}{\partial \Delta}(\Delta, x) &= (a + bx)\frac{\partial f}{\partial x}(\Delta, x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta, x) - \left(x^2 + d\right) f(\Delta, x) && (5.15) \\ f(0, x) &= g(x) && (5.16) \end{aligned}$$

where $g(x)$ is the final asset payoff, for example, $g(x) = 1$ for zero-coupon bonds. Conditional moments of the state variable are identical to those for the model of Vasicek (1977). Zero-coupon bond prices are also known in closed form, although the expression is quite tedious. However, in contrast to the Vasicek (1977) case, bond prices do have a singularity for some values of Δ; depending on the parameter values, there may be a singularity for a negative real value of Δ, or there may be a pair of singularities at complex conjugate values of Δ.

The pricing PDE has a form that is similar to that of the Vasicek (1977) model. When expressed in the form of Equation 4.10, the final term of the PDE is quadratic in the state variable, but the coefficients are

[4] We assume here a normalization of the state variable, by affine transformation, that makes the quadratic term in the interest rate function unity, and the linear function zero. Ahn, Dittmar, and Gallant (2002) permit the quadratic coefficient to be zero, in which case the model cannot be expressed in the form we use. However, in this case, the model is equiavelent to the model of Vasicek (1977). We therefore do not consider this possibility. Another possibility is that the quadratic coefficient is negative, but such a model would not be very realistic; for example, the interest rate could achieve arbitraily negative values, but would be bounded above. We do not consider this possibility either.

different than in the Vasicek (1977) case:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left[\frac{\left(\frac{a}{\sigma} + by\right)^2}{2} + \frac{b}{2} + (y\sigma)^2 + d\right] h(\Delta, y) \tag{5.17}$$

$$h(0, y) = e^{\frac{b}{2}\left(y + \frac{a}{b\sigma}\right)^2} g(y\sigma) \tag{5.18}$$

For a zero-coupon bond, the final condition is:

$$h(0, y) = e^{\frac{b}{2}\left(y + \frac{a}{b\sigma}\right)^2} \tag{5.19}$$

As in the Vasicek (1977) model, the payoff condition satisfies the conditions of Corollary 4 with:

$$\kappa = -\sqrt{b^2 + 2\sigma^2} \tag{5.20}$$

and therefore converges for at least some circle in τ. If such a circle includes $\tau = 1/\left(2\sqrt{b^2 + 2\sigma^2}\right)$, then a power series expansion in τ converges uniformly for values that correspond to the interval $\Delta \in [0, +\infty)$; if the singularities lie close to the center $\tau = 0$, then a power series expansion in τ will not converge on this interval, although in that case a second time change, through application of Theorem 3, can ensure uniform convergence on the entire interval $\Delta \in [0, +\infty)$. The location of the singularities depends on the values of the parameters.

We do not examine the accuracy of expansions to the first and second moments of the state variable, since these are the same as in the case of the previous section. However, the bond price implications of the two models are different. We choose parameter values $a = 0.090557$, $b = -0.5$, $\sigma = 0.079131$, and $d = 0.040936$. With these values, the interest rate process has the same unconditional mean and variance as in the case of the previous section, but has a kurtosis of 5 (since the interest rate is conditionally and unconditionally Gaussian in the case of Vasicek (1977), its kurtosis is then equal to 3). We take the initial value of the state variable to be $X_0 = 0.138072$, which corresponds to an initial interest rate of 6%. Figure 10 shows the price of a zero-coupon bond, as a function of maturity, using the true bond price expression, and a power series approximation in Δ. As shown, even with a large number of terms, the approximation begins to diverge badly for $\Delta \geq 5$. In fact, for these parameter values, there are singularities in the complex plane with modulus of approximately 5.28, so the power series actually diverges for values of Δ greater than this value.

As in the Vasicek (1977) case, we do not show the accuracy of a power series approximation in τ, because even with only one term, although slightly less accurate than in the Vasicek (1977) case, it is still sufficiently close that the two lines in the graph would appear to coincide. With only one term in the expansion, the largest relative error in the range $\Delta \in [0, 20)$ is approximately 0.0075; with a second term, the largest relative error declines to approximately 0.0000824, and with three terms, the largest relative error declines to 0.000000984. For the particular parameter values considered, the singularities in the pricing function correspond to values of τ that lie outside the smallest circle needed to include the point $\Delta = +\infty$; for other parameter values, this might not be the case, but in those cases, the range of convergence can be extended by application of Theorem 3. In all cases, uniform convergence on $\Delta \in [0, +\infty)$ is possible. As shown in the numeric examples, power

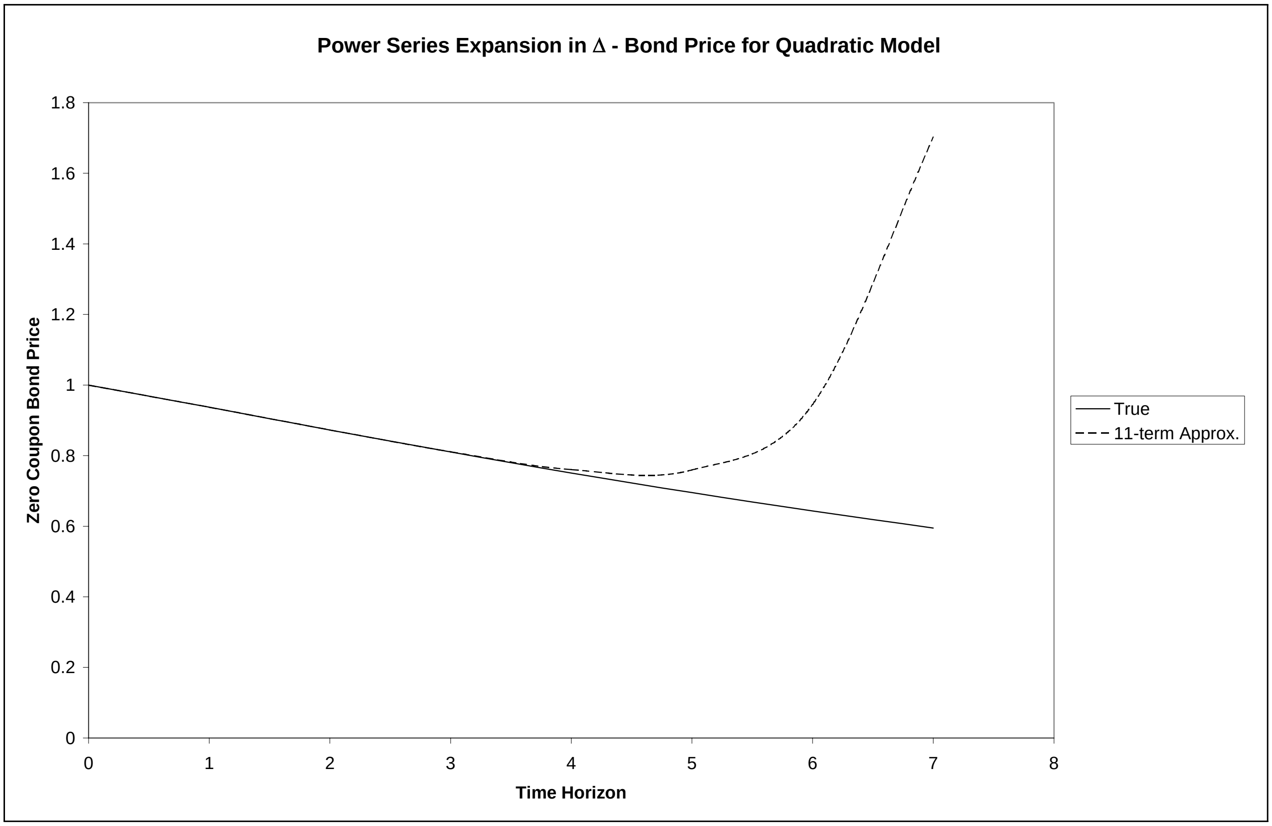

Figure 10: This figure shows the price of a zero-coupon bond in the model of Ahn, Dittmar, and Gallant (2002), using the closed form expression and an 11-term power series approximation in Δ. As shown, the 11-term approximation is quite accurate up to a time horizon of approximately 4 years, but diverges badly from the true value for longer horizons.

series expansions in τ appear to be extremely accurate, even with a very small number of terms.

5.3 Other Brownian Motion

In the previous two sections, we have examined the term structure models of Vasicek (1977) and of Ahn, Dittmar, and Gallant (2002), and found that the method of power series expansion after time transformation produces very accurate approximations to conditional moment functions and bond price functions. Of course, closed form expressions for these quantities are available for both models, so these two case serve as illustrative examples only. But it is easy to construct other models in which the bond pricing function can be found by series approximation. Both of the models considered previously had a pricing PDE which would be transformed to a canonical form:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left[b(y-\phi)^2 + d\right] h(\Delta, y) \tag{5.21}$$

Furthermore, the final condition for a zero-coupon bond price in each model is of the form:

$$h(0, y) = e^{a(y-\psi)^2 + w} \tag{5.22}$$

The region of analyticity of bond prices then follows from Corollary 4. However, there are many final conditions which satisfy the conditions of Corollary 4. Specifically, we can contemplate final conditions which are:

1. Polynomials
2. Exponential affine
3. Exponential quadratic
4. products of polynomials and exponential affine or quadratic functions
5. sums of the above functions

Every choice of a final condition corresponds to a term structure models. By choosing a final condition, we can reverse the changes of variables used to derive the standard form of the pricing PDE, to derive a new term structure model with analytic bond prices. Such bond prices can therefore be found through power series approximations, or possibly in closed-form. Consider the final condition:

$$h(0, y) = u(y) \tag{5.23}$$

Since the final payoff of a zero-coupon bond is simply equal to one, this final condition effectively specifies the change of dependent variable:

$$h(\Delta, y) = u(y) f(\Delta, y) \tag{5.24}$$

By substitution into the PDE, we find:

$$\frac{\partial f}{\partial \Delta}(\Delta, y) = \frac{\frac{\partial u}{\partial y}(y)}{u(y)} \frac{\partial f}{\partial y}(\Delta, y) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\Delta, y) - \left[b(y-\phi)^2 + d - \frac{1}{2}\frac{\frac{\partial^2 u}{\partial y^2}(y)}{u(y)} \right] f(\Delta, y) \tag{5.25}$$

with final condition:

$$f(0, y) = 1 \tag{5.26}$$

The function $f(\Delta, y)$ is therefore the bond price in a term structure model in which the state variable has the drift and diffusion indicated by the coefficients of the first two terms on the right-hand side of the PDE, and the interest rate is indicated by the coefficient of the last term. By this method, we can therefore construct a wide variety of term structure models in which bond prices can be approximated by power series approximations in τ, by application of Corollary 4, such that convergence is uniform on $\Delta \in [0, +\infty)$.

5.4 Cox, Ingersoll, and Ross

The interest rate and bond pricing model of Cox, Ingersoll, and Ross (1985), the instantaneous interest rate follows a Feller square-root process:

$$dX_t = (a + bX_t)\,dt + \sigma\sqrt{X_t}dW_t \tag{5.27}$$

In this case, the interest rate is equal to the state variable itself:

$$r(x) = x \tag{5.28}$$

The partial differential equation satisfied by bond (and other asset prices) is therefore:

$$\frac{\partial f}{\partial \Delta}(\Delta, x) = (a + bx)\frac{\partial f}{\partial x}(\Delta, x) + \frac{\sigma^2 x}{2}\frac{\partial^2 f}{\partial x^2}(\Delta, x) - xf(\Delta, x) \tag{5.29}$$

$$f(0, x) = g(x) \tag{5.30}$$

where $g(x)$ is the final asset payoff, for example, $g(x) = 1$ for zero-coupon bonds. The bond prices are also known in closed-form:

$$P(\Delta, x) = \exp\left[\begin{array}{c} \frac{2x}{b-\sqrt{b^2+2\sigma^2}\coth\left(\frac{\Delta}{2}\sqrt{b^2+2\sigma^2}\right)} \\ -\frac{a}{\sigma^2}\left(\begin{array}{c} b\Delta - \Delta\sqrt{b^2+2\sigma^2} + \\ 2\ln\left[\frac{1}{2}\left(1 - \frac{b}{\sqrt{b^2+2\sigma^2}}\right)\left(1 + \exp\left[\Delta\sqrt{b^2+2\sigma^2} + \frac{b}{\sigma^2}\left(b + \sqrt{b^2+2\sigma^2}\right)\right]\right)\right] \end{array} \right) \end{array} \right] \tag{5.31}$$

These bond prices are not entire, having a singularity at the point $\Delta = \ln\left[1 - \left(b^2 + b\sqrt{b^2+2\sigma^2}\right)/\sigma^2\right]/\sqrt{b^2+2\sigma^2}$. By making the change of variables to exporess the PDE in the form of Equation 4.10, however, the problem of bond pricing in the model of Cox, Ingersoll, and Ross (1985) is reduced to the general affine case. The conditions of Corollary 4 are satisfied for $\kappa = \sqrt{b^2+2\sigma^2}$, for sufficiently constrainted $k(\alpha)$. There exists a singularity in the complex plane in τ ; however, for reasonable values of the parameters for the US interest rate

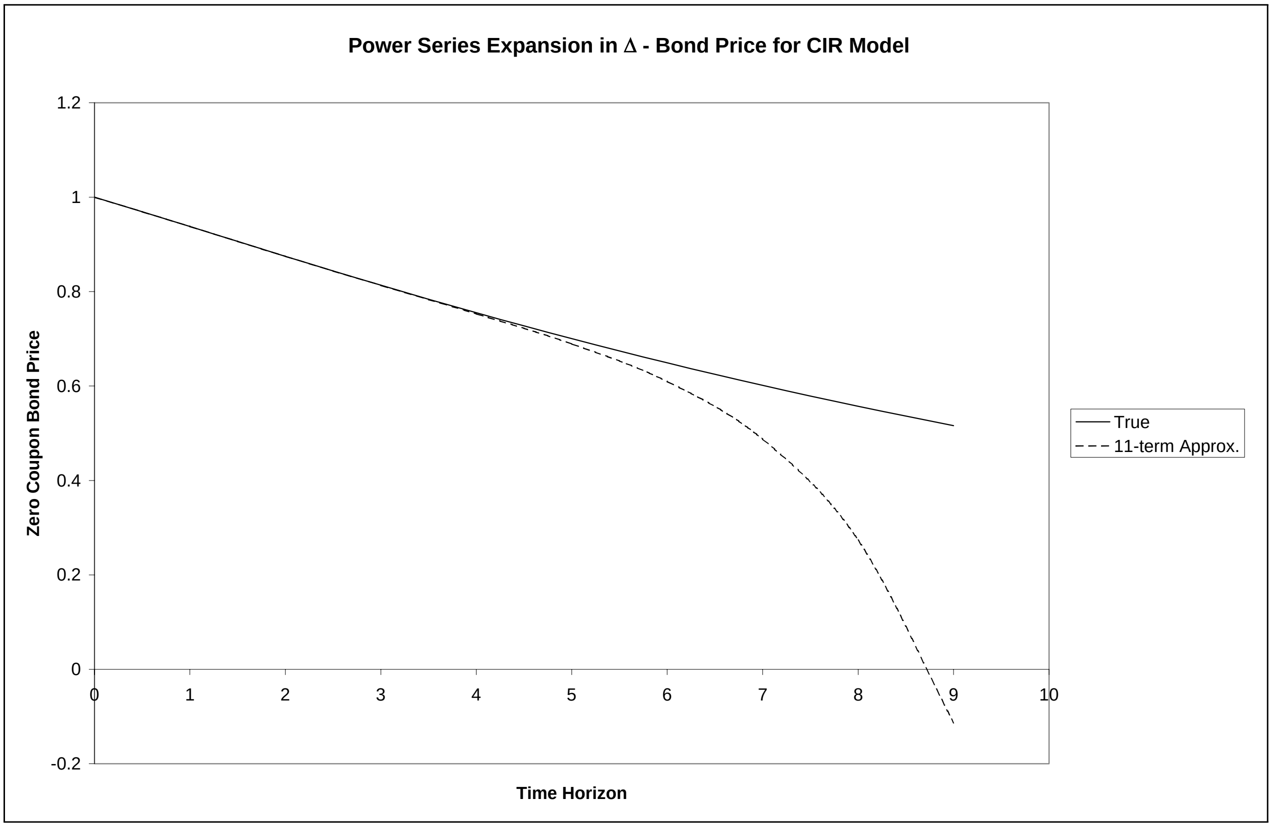

Figure 11: This figure shows the price of a zero-coupon bond in the model of Cox, Ingersoll, and Ross (1985), using the closed form expression and an 11-term power series approximation in Δ. As shown, the 11-term approximation is quite accurate up to a time horizon of approximately 4 years, but deviates badly from the true value for longer horizons. The series diverges for values of Δ larger than approximately 8.24.

process, this singularity lies well outside the cricle in τ needed to ensure uniform convergence of bond pirces on the interval $\Delta \in [0, +\infty)$. Consequently, a power series approximation of $w(\tau, z)$ in τ typically converges uniformly for all values corresponding to $\Delta \in [0, +\infty)$.

We now examine numerically the accuracy of two methods of power series expansion: directly in Δ, and in τ instead. We choose parameter values $a = 0.04$, $b = -0.5$, and $\sigma = 0.15$, which are roughly realistic values for a US interest rate process. With these parameter values, the bond price function has a singularity in the complex plane, with modules of approximately $|\Delta| = 8.24$. Therefore, a power series expansion in Δ does not converge for maturities longer than this value. Figure 11 shows the accuracy of a power series expansion in Δ to a zero-coupon bond price. As shown, the expansion is accurate for small values of Δ, but even after 11 terms, it deviates significantly for maturities beyond four or five years. The series diverges for maturities beyond approximately 8.24 years.

Unlike the cases of Vasicek (1977) and Ahn, Dittmar, and Gallant (2002), the difference between true bond prices and the expansions in τ is actually noticable graphically, when the number of terms is very small. Figure

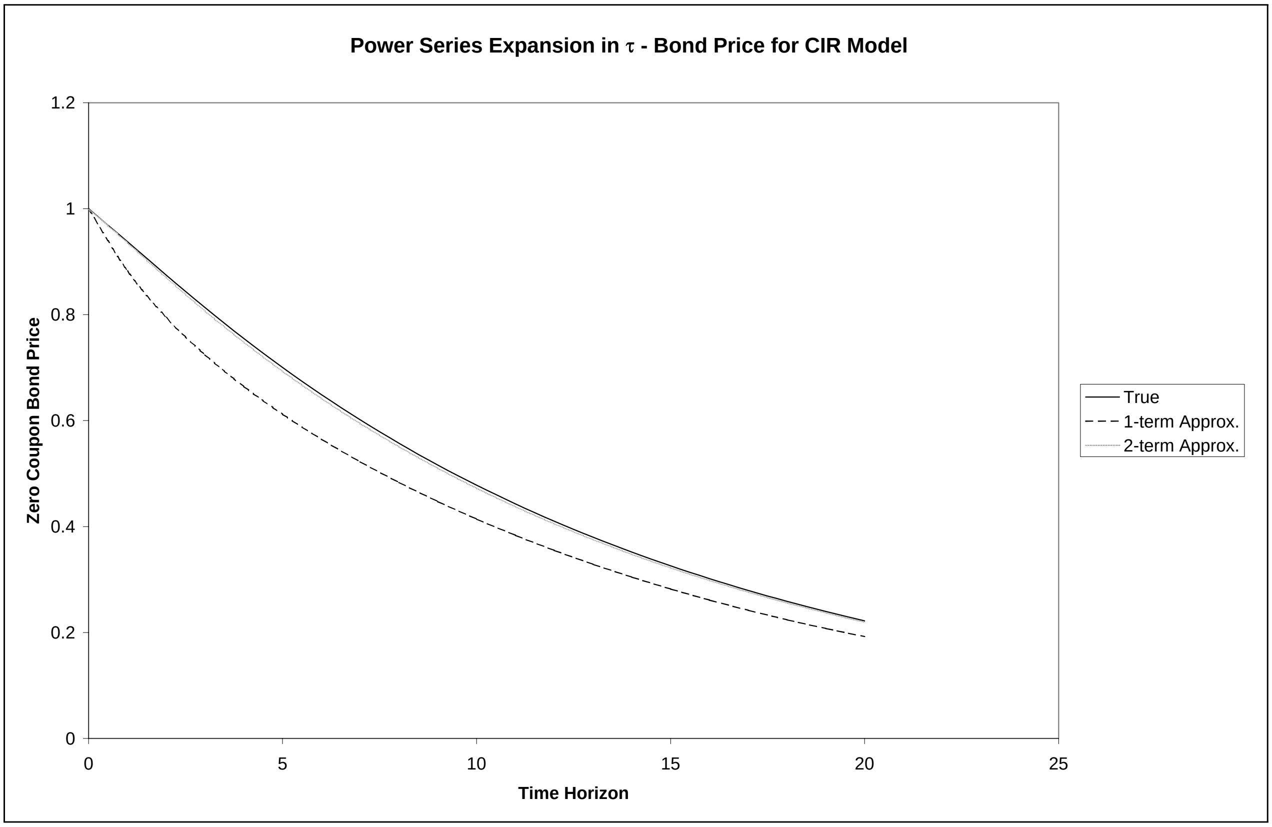

Figure 12: This figure shows the price of a zero-coupon bond in the model of Cox, Ingersoll, and Ross (1985), using the closed form expression and 1-term and 2-term power series approximations in τ. As shown, the 1-term approximation deviates noticably from the true price, but the 2-term approximation is much closer, for maturities to 20 years and longer. The series does not converge for all values of τ , but for the parameter values used, the converge is within a circle that corresponds to $\Delta \in [0, +\infty)$. Expansions with more terms (not shown) are not noticably different from the true price.

12 shows the accuracy of power series approximations in τ. As shown, an approximation with two terms is very accurate for maturities of up to 20 years; for longer maturities, the 2-term approximation is nearly as accurate. The relative error of the 20 year bond price for the 1-term expansion is approximately 0.1342; for a 2-term expansion, the relative error is approximately 0.0119. The two and three term expansions have relative errors of approximately 0.000870 and 0.000056, respectively, which correspond to errors in the yield of less than one basis point.

As in the previous cases, we find a power series expansion approach in τ leads to very close approximations even with a very small number of terms, for values of τ that correspond to large positive values of Δ. By contrast, power series expansions in Δ, although they do converge for any value of Δ, are quite inaccurate for large values of Δ, unless a very large (and impractical) number of terms are included in the approximation; furthermore, power series expansions in Δ diverge for sufficiently long maturites.

6 Conclusion

We have examined the problem of approximating conditional moments and bond prices for a wide variety of pricing problems. We have developed a method for determining the analyticity of conditional moments and bond prices in a neighborhood of the origin (in time) for a wide class of problems, establishing convergence of a power series (in time) for some time interval; we have further developed the method of time transformations to improve the convergence properties of these power series approximations; in many cases, uniform convergence for all positive maturities is possible. These methods make feasible rapid calculation of bond prices in many cases in which such calculation would otherwise not be practical.

Potential future work includes extension of the method to multivariate diffusions. Although some multivariate cases (e.g., independent state variable processes that enter additively into the interest rate function), in the general multivariate case, it is not even possible to express the pricing PDE in the canonical form in all cases. However, the method of change of dependent variable can undoubtedly lead to construction of new term structure models in which (after the change of variable) the pricing PDE is the same as the PDE that arises in multivariate affine models. Since polynomials of affine diffusions are analytic in time, at least a partial characterization of the final conditions with analytic moments is possible; as in the univariate case, each final condition with analytic moments corresponds to a term structure model with analytic bond prices. Such methods remain to be explored in full detail, however.

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7 Appendix

This appendix includes proofs of the theorems and corollaries in the main text. However, the following lemma will be helpful in many contexts.

Lemma 1. *Let $f(x)$ be an entire function, with power series representation:*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{7.1}$$

Then, for a given $k > 0$, there exists a $c > 0$ such that:

$$|f(x)| \leq c e^{\frac{|x|^2}{2k}} \tag{7.2}$$

if and only if the coefficients of the power series satisfy:

$$|a_n| \leq \frac{d}{k^{\frac{n}{2}} \sqrt{n!}} \tag{7.3}$$

for some $d > 0$ and for all $n \geq 0$. If these conditions are satisfied, then, for a given $k_1 < k$, there exists a $c_1 > 0$ such that:

$$|f'(x)| \leq c_1 e^{\frac{|x|^2}{2k_1}} \tag{7.4}$$

Proof:

Since $f(x)$ is entire, it can be represented by the Cauchy integral formula:

$$f(x) = \oint_C \frac{f(z)}{z - x} dz \tag{7.5}$$

where C is any closed curve such that x is inside the curve. The derivatives of are given by:

$$f^{(n)}(x) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - x)^{n+1}} dz \tag{7.6}$$

The coefficients of the power series are:

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} dz \tag{7.7}$$

If we take the curve C to be a circle with radius r, then:

$$|a_n| \leq \left| \frac{1}{2\pi i} \right| \left| \oint_C \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(z)|}{r^n} d\theta = \frac{c e^{\frac{r^2}{2k}}}{r^n} \tag{7.8}$$

This inequality imposes different bounds on the coefficients for different values of $r > 0$, but all of them hold for all n. We choose $r = \sqrt{kn}$, in which case the coefficients are bounded by:

$$|a_n| \leq c \left[\frac{e}{kn} \right]^{\frac{n}{2}} \leq \frac{d}{k^{\frac{n}{2}} \sqrt{n!}} \tag{7.9}$$

for a sufficiently large value of d. The other direction of the implication follows from:

$$|f(x)| \leq \sum_{n=0}^{\infty} |a_n| |x|^n = \sum_{n=0}^{\infty} \left| \frac{d}{k^{\frac{n}{2}} \sqrt{n!}} \right| |x|^n = d \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left(\frac{|x|}{\sqrt{k}} \right)^n \tag{7.10}$$

$$= d \sum_{n=0}^{\infty} \frac{1}{\sqrt{(2n)!}} \left(\frac{|x|}{\sqrt{k}} \right)^{2n} + d \sum_{n=0}^{\infty} \frac{1}{\sqrt{(2n+1)!}} \left(\frac{|x|}{\sqrt{k}} \right)^{2n+1} \leq c e^{\frac{|x|^2}{2k}} \tag{7.11}$$

for an appropriately chosen value of c.

From the power series representation of $f(x)$:

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \tag{7.12}$$

So if we define the coefficients of the derivative of $f(x)$ to be b_n, then:

$$b_n = (n+1) a_{n+1} \leq (n+1) \frac{d}{k^{\frac{n+1}{2}} \sqrt{(n+1)!}} = \sqrt{\frac{n+1}{k}} \frac{d}{k^{\frac{n}{2}} \sqrt{n!}} \leq \frac{d_1}{k_1^{\frac{n}{2}} \sqrt{n!}} \tag{7.13}$$

for appropriate values of d_1 and k_1. The boundedness condition on $f'(x)$ follows by applying the first part of the lemma with $f'(x)$ in place of $f(x)$.

QED

7.1 Proof of Theorem 1

Let $k \neq 0$ be a complex constant, and let $0 < r < 1$. If $f(\Delta, x)$ is defined and analytic in Δ in the region where:

$$-\arccos \sqrt{1-r^2} < \operatorname{Im}(k\Delta) < \arccos \sqrt{1-r^2} \tag{7.14}$$

$$-\ln \left(\frac{\cos[\operatorname{Im}(k\Delta)]}{+\sqrt{\cos^2[\operatorname{Im}(k\Delta)] - (1-r^2)}} \right) < \operatorname{Re}(k\Delta) < -\ln \left(\frac{\cos[\operatorname{Im}(k\Delta)]}{-\sqrt{\cos^2[\operatorname{Im}(k\Delta)] - (1-r^2)}} \right) \tag{7.15}$$

then the function $h(\tau, x) = f(\Delta, x)$ (where $\Delta = \Delta_k(\tau)$ is defined by Equation 3.9) is analytic in τ in the region $|\tau| < r$. Conversely, if $h(\tau, x)$ is defined and analytic in τ in the region $|\tau| < r$, then $f(\Delta, x) = h(\tau, x)$ (where $\tau = \tau_k(\Delta)$ is defined by Equation 3.2) is analytic in Δ in the region indicated by Equations 3.11 and 3.12.

Proof:

Consider the transformation of Equation *3.2*:

$$\tau_k(\Delta) = 1 - \exp(-k\Delta) \tag{7.16}$$

We note that $\tau_k(\Delta)$ is everywhere an analytic function of Δ. The inverse transformation is given by Equation *3.9*:

$$\Delta_k(\tau) = \frac{\ln(1-\tau)}{-k} \tag{7.17}$$

inherits the branch cut discontinuity of the logarithm function. Specifically, the principal branch of the logarithm function has a branch cut discontinuity along the entire negative real axis. This corresponds to values of τ satisfying:

$$\begin{aligned} \operatorname{Re}(\tau) &\geq 1 && (7.18)\\ \operatorname{Im}(\tau) &= 0 && (7.19)\end{aligned}$$

At all other values of τ , the inverse transformation is an analytic function of Δ.

We now consider the modulus of $\tau_k(\Delta)$:

$$\begin{aligned} |\tau_k(\Delta)| &= |1-\exp(-k\Delta)| && (7.20)\\ &= |1-\exp[-\operatorname{Re}(k\Delta)]\cos[\operatorname{Im}(k\Delta)] + i\exp[-\operatorname{Re}(k\Delta)]\sin[\operatorname{Im}(k\Delta)]| && (7.21)\\ &= \sqrt{1-2\exp[-\operatorname{Re}(k\Delta)]\cos[\operatorname{Im}(k\Delta)]+\exp[-2\operatorname{Re}(k\Delta)]} && (7.22)\end{aligned}$$

The values of Δ for which $\tau_k(\Delta)$ lies within a circle of radius r can then be found:

$$\begin{aligned} |\tau_k(\Delta)|^2 &< r^2 && (7.23)\\ 1-2\exp[-\operatorname{Re}(k\Delta)]\cos[\operatorname{Im}(k\Delta)]+\exp[-2\operatorname{Re}(k\Delta)] &< r^2 && (7.24)\\ \cos^2[\operatorname{Im}(k\Delta)]-2\exp[-\operatorname{Re}(k\Delta)]\cos[\operatorname{Im}(k\Delta)]+\exp[-2\operatorname{Re}(k\Delta)] &< r^2-1+\cos^2[\operatorname{Im}(k\Delta)] && (7.25)\\ (\exp[-\operatorname{Re}(k\Delta)]-\cos[\operatorname{Im}(k\Delta)])^2 &< r^2-1+\cos^2[\operatorname{Im}(k\Delta)] && (7.26)\end{aligned}$$

From the last line, it must be the case that:

$$\begin{aligned} r^2-1+\cos^2[\operatorname{Im}(k\Delta)] &> 0 && (7.27)\\ \cos^2[\operatorname{Im}(k\Delta)] &> 1-r^2 && (7.28)\end{aligned}$$

However, this condition is not sufficient for satisfaction of the original inequality. We must also have:

$$\begin{aligned} -\sqrt{r^2-1+\cos^2[\operatorname{Im}(k\Delta)]} &< \begin{pmatrix} \exp[-\operatorname{Re}(k\Delta)] \\ -\cos[\operatorname{Im}(k\Delta)]\end{pmatrix} < +\sqrt{r^2-1+\cos^2[\operatorname{Im}(k\Delta)]} && (7.29)\\ \begin{pmatrix} \cos[\operatorname{Im}(k\Delta)] \\ -\sqrt{r^2-1+\cos^2[\operatorname{Im}(k\Delta)]}\end{pmatrix} &< \exp[-\operatorname{Re}(k\Delta)] < \begin{pmatrix} \cos[\operatorname{Im}(k\Delta)] \\ +\sqrt{r^2-1+\cos^2[\operatorname{Im}(k\Delta)]}\end{pmatrix} && (7.30)\\ \ln\begin{pmatrix} \cos[\operatorname{Im}(k\Delta)] \\ -\sqrt{r^2-1+\cos^2[\operatorname{Im}(k\Delta)]}\end{pmatrix} &< -\operatorname{Re}(k\Delta) < \ln\begin{pmatrix} \cos[\operatorname{Im}(k\Delta)] \\ +\sqrt{r^2-1+\cos^2[\operatorname{Im}(k\Delta)]}\end{pmatrix} && (7.31)\\ -\ln\begin{pmatrix} \cos[\operatorname{Im}(k\Delta)] \\ +\sqrt{\cos^2[\operatorname{Im}(k\Delta)]+(1-r^2)}\end{pmatrix} &< \operatorname{Re}(k\Delta) < -\ln\begin{pmatrix} \cos[\operatorname{Im}(k\Delta)] \\ -\sqrt{\cos^2[\operatorname{Im}(k\Delta)]+(1-r^2)}\end{pmatrix} && (7.32)\end{aligned}$$

The values of Δ that satisfy Equations 7.28 and 7.32 are therefore a subset of those values of Δ that satisfy Equations 7.14 and 7.15.

Now, assume that $f(\Delta, x)$ is defined and analytic in the indicated region. Then $\tau_k(\Delta)$ is within a circle of radius r for values of Δ in this region, and the inverse transformation $\Delta_k(\tau)$ is analytic in τ in this region. The function $h(\tau, x) = f(\Delta_k(\tau), x)$ is therefore defined and analytic in τ, since it is an analytic function of an analytic function.

Conversely, assume that $h(\tau, x)$ is defined and analytic in $|\tau| < r$. Then the function $f(\Delta, x) = h(\tau_k(\Delta), x)$ is analytic in the indicated region of Δ, since $\tau_k(\Delta)$ is everywhere analytic in Δ, and $h(\tau, x)$ is analytic for values of τ which correspond to the indicated region of Δ.

QED

7.2 Proof of Corollary 1

Let $k \neq 0$ be a complex constant. If $f(\Delta, x)$ is defined and analytic in Δ in the region where:

$$-\frac{\pi}{2} < \mathrm{Im}(k\Delta) < \frac{\pi}{2} \tag{7.33}$$

$$\mathrm{Re}(k\Delta) > -\ln(2) - \ln(\cos[\mathrm{Im}(k\Delta)]) \tag{7.34}$$

then the function $h(\tau, x) = f(\Delta, x)$ (where $\Delta = \Delta_k(\tau)$ is defined by Equation 3.9) is analytic in τ in the region $|\tau| < 1$. Conversely, if $h(\tau, x)$ is defined and analytic in τ in the region $|\tau| < 1$, then $f(\Delta, x) = h(\tau, x)$ (where $\tau = \tau_k(\Delta)$ is defined by Equation 3.2) is analytic in Δ in the region indicated by Equations 3.13 and 3.14.

Proof:

From the proof of Theorem 1, we know that the restrictions on Δ given by Equations 7.33 and 7.34 are simply those of Equations 7.14 and 7.15 with the value of r approaching 1 from below (note that the upper limit on $\mathrm{Re}(k\Delta)$ approaches $+\infty$ in this case). Given analyticity of $f(\Delta, x)$ in Δ in the indicated region, for any point τ with $|\tau| < 1$, we can choose some $\varepsilon > 0$ such that $|\tau| < 1 - \varepsilon$. The conditions of Theorem 1 are then satisfied with $r = 1 - \varepsilon$, so $h(\tau, x)$ is analytic for this value of τ. Applying this procedure for every $|\tau| < 1$, we conclude that the function $h(\tau, x)$ is analytic everywhere within the unit circle in τ.

Conversely, suppose that $h(\tau, x)$ is analytic in τ in the region $|\tau| < 1$. Then for a given point Δ that satisfies Equations 7.33 and 7.34, it must be the case that Δ also satisfies Equations 7.14 and 7.15 for some $r = 1 - \varepsilon$. For this value of r, the conditions of Theorem 1 are satisfied, so $f(\Delta, x)$ is analytic for this value of Δ. Since this reasoning applies for every value of Δ satisfying Equations 7.33 and 7.34, $f(\Delta, x)$ must be analytic within this region.

QED

7.3 Proof of Theorem 2

Let $k \neq 0$ be a complex constant, and let $r > 1$. If $h(\tau, x)$ is defined and analytic within the circle $|\tau| < r$, with $r > 1$, then $f(\Delta, x) = h(\tau, x)$ (where $\tau = \tau_k(\Delta)$ is defined by Equation 3.2) is defined and analytic in the region where:

$$\mathrm{Re}(k\Delta) > -\ln\left(\cos[\mathrm{Im}(k\Delta)] + \sqrt{\cos^2[\mathrm{Im}(k\Delta)] + r^2 - 1}\right) \tag{7.35}$$

Furthermore, $f(\Delta, x)$ *is periodic in this region with period $2\pi i/k$, and tends to a limit as* $\mathrm{Re}(k\Delta) \to +\infty$.

Proof:

From the Proof of Theorem , we have the following condition for $|\tau_k(\Delta)| < r$:

$$(\exp[-\mathrm{Re}(k\Delta)] - \cos[\mathrm{Im}(k\Delta)])^2 < r^2 - 1 + \cos^2[\mathrm{Im}(k\Delta)] \tag{7.36}$$

This relation remains valid for $r > 1$. However, in this case, the right-hand side is always strictly positive. Consequently, the only requirement for $|\tau_k(\Delta)| < r$ with $r > 1$ is:

$$-\sqrt{r^2 - 1 + \cos^2[\mathrm{Im}(k\Delta)]} < \begin{pmatrix} \exp[-\mathrm{Re}(k\Delta)] \\ -\cos[\mathrm{Im}(k\Delta)] \end{pmatrix} < \sqrt{r^2 - 1 + \cos^2[\mathrm{Im}(k\Delta)]} \tag{7.37}$$

$$\begin{pmatrix} \cos[\mathrm{Im}(k\Delta)] \\ -\sqrt{r^2 - 1 + \cos^2[\mathrm{Im}(k\Delta)]} \end{pmatrix} < \exp[-\mathrm{Re}(k\Delta)] < \begin{pmatrix} \cos[\mathrm{Im}(k\Delta)] \\ +\sqrt{r^2 - 1 + \cos^2[\mathrm{Im}(k\Delta)]} \end{pmatrix} \tag{7.38}$$

Note that the left expression in this last chain of inequalities is always negative for $r > 1$; since exponential functions of real quantities are always positive, this inequality is always satisfied. So the requirement for $|\tau_k(\Delta)| < r$ is:

$$\exp[-\mathrm{Re}(k\Delta)] < \cos[\mathrm{Im}(k\Delta)] + \sqrt{r^2 - 1 + \cos^2[\mathrm{Im}(k\Delta)]} \tag{7.39}$$

$$-\mathrm{Re}(k\Delta) < \ln\left(\cos[\mathrm{Im}(k\Delta)] + \sqrt{r^2 - 1 + \cos^2[\mathrm{Im}(k\Delta)]}\right) \tag{7.40}$$

$$\mathrm{Re}(k\Delta) > -\ln\left(\cos[\mathrm{Im}(k\Delta)] + \sqrt{r^2 - 1 + \cos^2[\mathrm{Im}(k\Delta)]}\right) \tag{7.41}$$

So if Δ satisfies this last restriction, then $|\tau_k(\Delta)| < r$, and the function $h(\tau, x)$ is by assumption defined and analytic in τ at $\tau = \tau_k(\Delta)$. Since $\tau_k(\Delta)$ is everywhere analytic in Δ, the function $f(\Delta, x) = h(\tau_k(\Delta), x)$ is therefore analytic in Δ at this point. Therefore, $f(\Delta, x)$ is defined and analytic for every value of Δ satisfying the inequality in the Theorem statement. Periodicity with period $2\pi i/k$ follows immediately from the definition of $\tau_k(\Delta)$. Finally, we note that as $\mathrm{Re}(k\Delta) \to +\infty$, $\tau_k(\Delta) \to 1$, and by the continuity of $h(\tau, x)$ in τ within the circle $|\tau| < r$ for some $r > 1$, we have $f(\Delta, x) \to h(1, x)$ as $\mathrm{Re}(k\Delta) \to +\infty$.

QED

7.4 Proof of Theorem 3

Let $g(y)$ be an entire function, let $c > 0$ be a positive constant, and let $k(\alpha)$ be a positive real continuous function defined on $\alpha \in [0, \pi]$ with $k(0) = k(\pi)$, such that:

$$\left|g\left(se^{i\alpha}\right)\right| \leq ce^{\frac{s^2}{2k(\alpha)}} \tag{7.42}$$

for all real s and $\alpha \in [0, \pi]$. Then there exists a function $h(\Delta, y)$ such that:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) \tag{7.43}$$

$$h(0, y) = g(y) \tag{7.44}$$

that is defined and analytic for all complex Δ and y such that $\Delta = re^{i\theta}$ with $\theta \in [0, 2\pi]$ and $0 \leq r < k\left(\frac{\theta}{2}\right)$.

Proof:

We show that the solution is given by:

$$h(\Delta, y) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} u\left(y, z\sqrt{\Delta}\right) dz \tag{7.45}$$

where the integral is an ordinary Lebesgue integral over the real line, and the function $u(y,x)$ is given by:

$$u(y,x) = \frac{g(y+x)+g(y-x)}{2} \tag{7.46}$$

and, for $\Delta \neq 0$, either square root may be chosen (note that the value of $u\left(y, z\sqrt{\Delta}\right)$ does not depend on the choice). It must be demonstrated that $h(\Delta,y)$ exists (i.e., that the integral converges), that $h(\Delta,y)$ is differentiable in Δ and y, and that it solves the PDE with final condition.

To show the existence of $h(\Delta,y)$ in the specified region, we first note that the function $u\left(y, z\sqrt{\Delta}\right)$ satisfies the boundedness condition:

$$\left|u\left(y, z\sqrt{\Delta}\right)\right| \leq de^{\frac{z^2 r}{2k\left(\frac{\theta}{2}\right)}} \tag{7.47}$$

for some $d > 0$, where $\Delta = re^{i\theta}$. The integrand in Equation 7.45 therefore satisfies the boundedness condition:

$$\left|\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} u\left(y, z\sqrt{\Delta}\right)\right| = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\left|u\left(y, z\sqrt{\Delta}\right)\right| \leq d\frac{e^{\frac{z^2}{2k\left(\frac{\theta}{2}\right)}\left[r-k\left(\frac{\theta}{2}\right)\right]}}{\sqrt{2\pi}} \tag{7.48}$$

Therefore, for any value of $r < k\left(\frac{\theta}{2}\right)$, the integrand is bounded and approaches zero in the tails sufficiently fast that the integral converges, and $h(\Delta,y)$ is therefore well-defined.

To show analyticity in Δ and y, we first note some properties of $u\left(y, z\sqrt{\Delta}\right)$. Since, by assumption, $g(y)$ is entire, it has a power series expansion (around any point y_0) that converges in the entire complex plane:

$$g(y) = \sum_{i=0}^{\infty} d_i(y_0)(y-y_0)^i \tag{7.49}$$

The function defined by:

$$u(y_0,x) = \frac{g(y_0+x)+g(y_0-x)}{2} \tag{7.50}$$

is therefore also entire in x, and has a power series representation given by:

$$u(y_0,x) = \sum_{i=0}^{\infty} d_{2i}(y_0)\,x^{2i} = \sum_{i=0}^{\infty} d_{2i}(y_0)\left(x^2\right)^i \tag{7.51}$$

The function $u(y_0,x)$ is therefore even in its second argument, so we can define:

$$v(y_0,w) = u\left(y_0, \sqrt{w}\right) \tag{7.52}$$

and the definition is unambiguous, since the right-hand-side has the same value no matter which square root is chosen. Furthermore, $v(y_0,w)$ is analytic in w. Thus, $v\left(y, z^2\Delta\right) = u\left(y, z\sqrt{\Delta}\right)$ is an analytic function of $z^2\Delta$. We can therefore write the proposed solution as:

$$h(\Delta,y) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} v\left(y, z^2\Delta\right) dz \tag{7.53}$$

The integrand is clearly analytic in Δ, y, and z^2. Furthermore, for all values of Δ and y satisfying the conditions of the theorem statement, the integrand is uniformly bounded in some neighborhood of Δ and y by a function of the form $\exp\left(-kz^2\right)$ for some $k > 0$. The integral itself is therefore also analytic in Δ and y at this point.

Finally, we show that $h(\Delta, y)$ satisfies the partial differential equation with final condition. Satisfaction of the final condition is straightforward; for $\Delta = 0$, we have:

$$h(0,y) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} u(y,0)\, dz = \int_{-\infty}^{+\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} g(y)\, dz = g(y) \int_{-\infty}^{+\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz = g(y) \tag{7.54}$$

Satisfaction of the partial differential equation can be demonstrated through integration by parts:

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) &= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} g''\left(y + z\sqrt{\Delta}\right) dz && (7.55)\\
&= \frac{e^{-\frac{z^2}{2}}}{2\sqrt{2\pi}\sqrt{\Delta}} g'\left(y + z\sqrt{\Delta}\right)\Bigg|_{-\infty}^{+\infty} + \frac{1}{2}\int_{-\infty}^{+\infty} \frac{z}{\sqrt{\Delta}} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} g'\left(y + z\sqrt{\Delta}\right) dz && (7.56)\\
&= \int_{-\infty}^{+\infty} \frac{z}{2\sqrt{\Delta}} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} g'\left(y + z\sqrt{\Delta}\right) dz && (7.57)\\
&= \frac{\partial h}{\partial \Delta}(\Delta, y) && (7.58)
\end{aligned}$$

QED

7.5 Proof of Corollary 2

Let $g(y)$ be an entire function, and for each positive real $k > 0$, let there be a positive real constant $c_k > 0$ such that:

$$\left|g\left(se^{i\alpha}\right)\right| \leq c_k e^{\frac{s^2}{2k}} \tag{7.59}$$

for all real s and all $\alpha \in [0, \pi]$. Then there exists a function $h(\Delta, y)$ such that:

$$\begin{aligned}
\frac{\partial h}{\partial \Delta}(\Delta, y) &= \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) && (7.60)\\
h(0, y) &= g(y) && (7.61)
\end{aligned}$$

that is defined and analytic for all complex Δ and y.

Proof:

Choose a value of k, and set $k(\alpha) = k$. Then the function $g(y)$ satisfies the growth condition of Theorem 3 for this $k(\alpha)$ and for some $c = c_k$. So by application of Theorem 3, there exists a solution to the partial differential equation (with final condition) $h(\Delta, y)$ that is analytic for all y and for all $|\Delta| < k$. Since we can choose any $k > 0$, the circle of analyticity can be shown to be as large as desired. Furthermore, the construction from Theorem 3 does not depend on c or $k(\alpha)$, so it is clear that the functions thus constructed are the same. Consequently, the solution is entire in Δ.

QED

7.6 Proof of Corollary 3

Let $g(y)$ be an entire function, let $c > 0$ be a positive constant, and let $k(\alpha)$ be a positive real continuous function defined on $\alpha \in [0, \pi]$ with $k(0) = k(\pi)$, such that:

$$\left|g\left(se^{i\alpha}\right)\right| \leq c e^{\frac{r^2}{2k(\alpha)}} \tag{7.62}$$

for all real s and $\alpha \in [0,\pi]$. Then there exists a function $h(\Delta, y)$ such that:

$$\begin{aligned}
\frac{\partial h}{\partial \Delta}(\Delta, y) &= \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) + (\phi y + \varphi)\, h(\Delta, y) && (7.63)\\
h(0, y) &= g(y) && (7.64)
\end{aligned}$$

that is defined and analytic for all complex Δ and y such that $\Delta = re^{i\theta}$ with $\theta \in [0, 2\pi]$ and $0 \leq r < k\left(\frac{\theta}{2}\right)$.

Proof:

We first consider the problem:

$$\begin{aligned}
\frac{\partial w}{\partial \Delta}(\Delta, z) &= \frac{1}{2}\frac{\partial^2 w}{\partial z^2}(\Delta, z) && (7.65)\\
w(0, z) &= g(z) && (7.66)
\end{aligned}$$

By application of Theorem 3, there exists a solution that is analytic in z and in Δ in the region indicated by the condition in the Corollary. We now define:

$$\begin{aligned}
z(\Delta, y) &= y + \frac{\phi}{2}\Delta^2 && (7.67)\\
h(\Delta, y) &= e^{\frac{\phi^2}{6}\Delta^3 + (\varphi + \phi y)\Delta} w(\Delta, z(\Delta, y)) && (7.68)
\end{aligned}$$

These functions are defined for all values of y, and for all values of Δ where $w(\Delta, z)$ is defined, and are clearly analytic in both variables in this region. Furthermore, the relation is invertible:

$$\begin{aligned}
y(\Delta, z) &= z - \frac{\phi}{2}\Delta^2 && (7.69)\\
w(\Delta, z) &= e^{\frac{\phi^2}{3}\Delta^3 - (\phi z + \varphi)\Delta} h(\Delta, y(\Delta, z)) && (7.70)
\end{aligned}$$

Plugging these last two expressions into the PDE specified by Equation 7.65, we find that $h(\Delta, y)$ satisfies the PDE in the Corollary statement. Furthermore, when $\Delta = 0$, from the definition of $y(\Delta, z)$ we have $y(0, z) = z$, and then from the definition of $w(\Delta, z)$, we have $w(0, z) = h(0, z) = g(z)$, so $h(\Delta, y)$ also satisfies the final condition in the Corollary statement.

QED

7.7 Proof of Corollary 4

Let $g(y)$ be an entire function, let $c > 0$ be a positive constant, let $\kappa \neq 0$ be any non-zero complex number, and let $k(\alpha)$ be a positive real continuous function defined on $\alpha \in [0, \pi]$, such that:

$$\left| e^{-\frac{\kappa}{2}s^2 e^{2i\alpha}} g\left(se^{i\alpha}\right) \right| \leq ce^{\frac{s^2}{2k(\alpha)}} \qquad (7.71)$$

for all real s and $\alpha \in [0, \pi]$. Then there exists a function $w(\tau, z)$ such that:

$$\begin{aligned}
\frac{\partial w}{\partial \tau}(\tau, z) &= \frac{1}{2}\frac{\partial^2 w}{\partial z^2}(\tau, z) && (7.72)\\
w(0, z) &= e^{-\frac{\kappa}{2}(z-\phi)^2} g(z) && (7.73)
\end{aligned}$$

that is analytic for all complex τ and z such that $\tau = re^{i\theta}$ with $\theta \in [0, 2\pi]$ and $0 \leq r < k\left(\frac{\theta}{2}\right)$. Furthermore, the function $h(\Delta, y)$ defined by:

$$h(\Delta, y) = e^{\frac{\kappa}{2}(y-\phi)^2 - \left(\varphi - \frac{\kappa}{2}\right)\Delta} w\left(\frac{e^{2\kappa\Delta} - 1}{2\kappa}, \phi + e^{\kappa\Delta}(y - \phi)\right) \tag{7.74}$$

satisfies:

$$\begin{aligned} \frac{\partial h}{\partial \Delta}(\Delta, y) &= \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left[\frac{\kappa^2}{2}(y - \phi)^2 + \varphi\right] h(\Delta, y) && (7.75) \\ h(0, y) &= g(y) && (7.76) \end{aligned}$$

for all complex Δ and y such that:

$$\frac{e^{2\kappa\Delta} - 1}{2\kappa} = re^{i\theta} \tag{7.77}$$

for some r and θ, with $\theta \in [0, 2\pi]$ and $0 \leq r < k\left(\frac{\theta}{2}\right)$.

Proof:

We first consider the problem:

$$\begin{aligned} \frac{\partial w}{\partial \tau}(\tau, z) &= \frac{1}{2}\frac{\partial^2 w}{\partial z^2}(\tau, z) && (7.78) \\ w(0, z) &= e^{-\frac{\kappa}{2}(z-\phi)^2} g(z) && (7.79) \end{aligned}$$

By application of Theorem 3, there exists a solution that is analytic in z and in τ in the region indicated by the condition in the Corollary. We now define:

$$\begin{aligned} \tau(\Delta) &= \frac{e^{2\kappa\Delta} - 1}{2\kappa} && (7.80) \\ z(\Delta, y) &= \phi + e^{\kappa\Delta}(y - \phi) && (7.81) \\ h(\Delta, y) &= e^{\frac{\kappa}{2}(y-\phi)^2 + \left(\frac{\kappa}{2} - \varphi\right)\Delta} w(\tau(\Delta), z(\Delta, y)) && (7.82) \end{aligned}$$

These functions are defined for all values of y, and for all values of Δ such that $\tau(\Delta)$ falls within the indicated region. These equations have a local inverse:

$$\begin{aligned} \Delta(\tau) &= \frac{\ln(1 + 2\kappa\tau)}{2\kappa} && (7.83) \\ y(\tau, z) &= \phi + \frac{(z - \phi)}{\sqrt{1 + 2\kappa\tau}} && (7.84) \\ w(\tau, z) &= e^{-\frac{\kappa}{2}\left(\frac{(z-\phi)^2}{1+2\kappa\tau}\right) - \left(\frac{\kappa}{2} - \varphi\right)\frac{\ln(1+2\kappa\tau)}{2\kappa}} h(\Delta(\tau), y(\tau, z)) && (7.85) \end{aligned}$$

Note that it does not matter which logarithm or which square root is chosen in the above equations. Plugging this expression for $w(\tau, z)$ into Equation 7.78, we find that $h(\Delta, y)$ solves the PDE in the Corollary statement. Furthermore, when $\Delta = 0$, from the definitions of $\tau(\Delta)$ and $y(\tau, z)$ we have $y(0, z) = z$, and then from the definition of $w(\tau, z)$, we have $w(0, z) = h(0, z) = g(z)$, so $h(\Delta, y)$ also satisfies the final condition in the Corollary statement.

QED

7.8 Proof of Theorem 4

Let $w_1(y)$ *and* $w_2(y)$ *be entire functions, let* $c>0$ *be a positive constant, and let* $k(\alpha)$ *be a positive real continuous function defined on* $\alpha\in[0,\pi]$ *with* $k(0)=k(\pi)$*, such that:*

$$\begin{aligned}\left|w_1\left(s^2e^{2i\alpha}\right)\right| &\leq ce^{\frac{s^2}{2k(\alpha)}} &(7.86)\\ \left|w_2\left(s^2e^{2i\alpha}\right)\right| &\leq ce^{\frac{s^2}{2k(\alpha)}} &(7.87)\end{aligned}$$

for all real s *and* $\alpha\in[0,\pi]$*. Then there exists a solution to the problem:*

$$\begin{aligned}\frac{\partial h}{\partial\Delta}(\Delta,y) &= \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta,y)+\frac{a}{y^2}h(\Delta,y) &(7.88)\\ h(0,y) &= g(y) &(7.89)\end{aligned}$$

where:

$$\begin{aligned}g(y) &= w_1\left(y^2\right)y^{\frac{1-\sqrt{1-8a}}{2}}+w_2\left(y^2\right)y^{\frac{1+\sqrt{1-8a}}{2}},\ \frac{\sqrt{1-8a}}{2}\notin\mathbb{N} &(7.90)\\ g(y) &= w_1\left(y^2\right)y^{\frac{1-\sqrt{1-8a}}{2}}+w_2\left(y^2\right)y^{\frac{1+\sqrt{1-8a}}{2}}\ln y,\ \frac{\sqrt{1-8a}}{2}\in\mathbb{N} &(7.91)\end{aligned}$$

that is defined and analytic for all complex Δ *and* $y\neq 0$ *such that* $\Delta=re^{i\theta}$ *for some* $\theta\in[0,2\pi]$ *and* $0\leq r<k\left(\frac{\theta}{2}\right)$*. Note that a branch cut discontinuity in* y *is possible.*

Proof:

Although the theorem statement is similar to that of Theorem 3, the proof is much more complicated. We therefore begin with an auxiliary lemma:

Lemma 2. *Let* $\psi(y)$ *be an entire function, let* $c>0$ *be a positive constant, and let* $k(\alpha)$ *be a positive real continuous function defined on* $\alpha\in[0,\pi]$ *with* $k(0)=k(\pi)$*, such that:*

$$\left|\psi\left(s^2e^{2i\alpha}\right)\right|\leq ce^{\frac{s^2}{2k(\alpha)}} \qquad (7.92)$$

for all real s *and* $\alpha\in[0,\pi]$*. Then there exists a function* $w(\Delta,x)$ *such that:*

$$\begin{aligned}\frac{\partial w}{\partial\Delta}(\Delta,y) &= \frac{\gamma}{y}\frac{\partial w}{\partial y}(\Delta,y)+\frac{1}{2}\frac{\partial^2 w}{\partial y^2}(\Delta,x) &(7.93)\\ w(0,y) &= \psi\left(y^2\right) &(7.94)\end{aligned}$$

that is defined and analytic for all complex Δ *and* y *such that* $\Delta=re^{i\theta}$ *for some* $\theta\in[0,2\pi]$ *and* $0\leq r<k\left(\frac{\theta}{2}\right)$*.*

Proof: See below.

Lemma 2 expresses a result for a PDE that is not in the canonical form developed by 4.10. However, it can be turned into the canonical form by a simple change of dependent variable, and we can express the solution to the problem in the theorem statement in terms of the solution to the problem of the Lemma, by reversing the change of variables.

We begin by applying Lemma 2 with $\gamma=\frac{1-\sqrt{1-8a}}{2}$ and $u(y)=w_1\left(y^2\right)$, and finding a solution $h_1(\Delta,y)=w(\Delta,y)$. By applying Lemma 2 again with $\gamma=\frac{1+\sqrt{1-8a}}{2}$ and $u(y)=w_2\left(y^2\right)$, we find another solution $h_2(\Delta,y)=w(\Delta,y)$. But now we can construct a solution to the original PDE:

$$h(\Delta,y)=y^{\frac{1-\sqrt{1-8a}}{2}}h_1(\Delta,y)+y^{\frac{1+\sqrt{1-8a}}{2}}h_2(\Delta,y) \qquad (7.95)$$

By inspection, this solution satisfies the PDE with final condition in the theorem statement; the theorem is now proven for the case $\frac{\sqrt{1-8a}}{2} \notin \mathbb{N}$.

QED

7.9 Proof of Lemma 2

Let $\psi(y)$ be an entire function, let $c > 0$ be a positive constant, and let $k(\alpha)$ be a positive real continuous function defined on $\alpha \in [0,\pi]$ with $k(0) = k(\pi)$, such that:

$$\left|\psi\left(s^2 e^{2i\alpha}\right)\right| \leq c e^{\frac{s^2}{2k(\alpha)}} \tag{7.96}$$

for all real s and $\alpha \in [0,\pi]$. Then there exists a function $w(\Delta, y)$ such that:

$$\frac{\partial w}{\partial \Delta}(\Delta, y) = \frac{\gamma}{y}\frac{\partial w}{\partial y}(\Delta, y) + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}(\Delta, y) \tag{7.97}$$

$$w(0, y) = \psi\left(y^2\right) \tag{7.98}$$

that is defined and analytic for all complex Δ and y such that $\Delta = re^{i\theta}$ for some $\theta \in [0, 2\pi]$ and $0 \leq r < k\left(\frac{\theta}{2}\right)$.

Proof:

We construct the solution explicitly, by the paramatrix method of Levi (1907), as described in Friedman (1964). Note, however, the assumptions of Friedman (1964) are not satisfied in this case, because the coefficients of the PDE are not bounded. We therefore must modify the method.

The solution is given by:

$$\begin{aligned} w(\Delta, y) &= \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\frac{\psi\left(y + u\sqrt{\Delta}\right) + \psi\left(y - u\sqrt{\Delta}\right)}{2}\right] du \\ &\quad + \int_0^1 \int_{-\infty}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \Delta \sum_{i=1}^{\infty} \left[\frac{\begin{array}{c}\Phi_i\left(v\Delta, y + u\sqrt{\Delta(1-v)}\right) \\ +\Phi_i\left(v\Delta, y - u\sqrt{\Delta(1-v)}\right)\end{array}}{2}\right] dudv \end{aligned}$$

where:

$$\phi_1(\Delta, y) = \gamma \int_{-\infty}^{+\infty} \frac{u^2 e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\frac{\psi\left(y + u\sqrt{\Delta}\right) - \psi\left(y - u\sqrt{\Delta}\right)}{2yu\sqrt{\Delta}}\right] du \tag{7.99}$$

$$\phi_{i+1}(\Delta, y) = \gamma\Delta \int_0^1 \int_{-\infty}^{+\infty} \frac{u^2 e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left[\frac{\begin{array}{c}\Phi_i\left(v\Delta, y + u\sqrt{\Delta(1-v)}\right) \\ -\Phi_i\left(v\Delta, y - u\sqrt{\Delta(1-v)}\right)\end{array}}{2yu\sqrt{\Delta(1-v)}}\right] dudv,\ i \geq 1 \tag{7.100}$$

Three things must be proven. First, it must be shown that the solution specified above exists (i.e., that the integrals and the infinite sum converge). Second, it must be shown that this solution is analytic in Δ and y. Finally, it must be shown that it solves the PDE and initial condition.

We beginning by proving several properties of the $\Phi_i(\Delta, y)$ functions. Beginning with $\Phi_1(\Delta, y)$, we note that the function:

$$\alpha(y, u, \Delta) = \frac{\psi\left(y + u\sqrt{\Delta}\right) - \psi\left(y - u\sqrt{\Delta}\right)}{2u\sqrt{\Delta}} \tag{7.101}$$

is an analytic function of y, u, and Δ, and furthermore, is even in u. By assumption, the integrand in the definition of $\phi_1(\Delta, y)$ is uniformly bounded by a function of the type $\exp\left(-ku^2\right)$ for some $k > 0$. Consequently, the integral exists, and is analytic in both y and Δ. We further note that the integrand (and therefore the integral) is even in y; this result follows immediately from the assumption that ψ is even in its argument. The function $\phi_1(\Delta, y)$ therefore exists, is analytic in both variables, and is even in y. Furthermore, it satisfies a boundedness condition:

$$
\begin{aligned}
|\phi_1(\Delta, y)| &\leq |\gamma| \int_{-\infty}^{+\infty} \frac{u^2 e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left| \frac{\psi\left(y + u\sqrt{\Delta}\right) - \psi\left(y - u\sqrt{\Delta}\right)}{2yu\sqrt{\Delta}} \right| du && (7.102) \\
&\leq |\gamma| \int_{-\infty}^{+\infty} \frac{u^2 e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} d e^{\frac{|y|^2 + u^2 r}{2k\left(\frac{\theta}{2}\right)}} du && (7.103) \\
&= d\,|\gamma|\, e^{\frac{|y|^2}{2k\left(\frac{\theta}{2}\right)}} \left[\frac{2k\left(\frac{\theta}{2}\right)}{k\left(\frac{\theta}{2}\right) - r}\right]^{\frac{3}{2}} && (7.104)
\end{aligned}
$$

where $\Delta = re^{i\theta}$.

Now, we note that, if $\Phi_i(\Delta, y)$ is an analytic function of Δ and y, then the two functions:

$$
\begin{aligned}
\alpha(\Delta, y, u, v) &= \left[\frac{\Phi_i\left(v\Delta, y + u\sqrt{\Delta(1-v)}\right) - \Phi_i\left(v\Delta, y - u\sqrt{\Delta(1-v)}\right)}{2u\sqrt{\Delta(1-v)}}\right] && (7.105) \\
\beta(\Delta, y, u, v) &= \left[\frac{\Phi_i\left(v\Delta, y + u\sqrt{\Delta(1-v)}\right) + \Phi_i\left(v\Delta, y - u\sqrt{\Delta(1-v)}\right)}{2}\right] && (7.106)
\end{aligned}
$$

are each analytic functions of u, v, Δ and y. Furthermore, $\alpha(\Delta, y, u, v)$ is odd in u, and $\beta(\Delta, y, u, v)$ is even in u.

QED

7.10 Proof of Corollary 5

Let $w_1(y)$ and $w_2(y)$ be entire functions, and for each positive real $k > 0$, let there be a positive real constant $c_k > 0$ such that:

$$
\begin{aligned}
\left|w_1\left(s^2 e^{2i\alpha}\right)\right| &\leq c_k e^{\frac{s^2}{2k}} && (7.107) \\
\left|w_2\left(s^2 e^{2i\alpha}\right)\right| &\leq c_k e^{\frac{s^2}{2k}} && (7.108)
\end{aligned}
$$

for all real s and all $\alpha \in [0, \pi]$. Then there exists a function $h(\Delta, y)$ such that:

$$
\begin{aligned}
\frac{\partial h}{\partial \Delta}(\Delta, y) &= \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) + \frac{a}{y^2} h(\Delta, y) && (7.109) \\
h(0, y) &= g(y) && (7.110)
\end{aligned}
$$

where:

$$g(y) = w_1(y^2)\, y^{\frac{1-\sqrt{1-8a}}{2}} + w_2(y^2)\, y^{\frac{1+\sqrt{1-8a}}{2}}, \quad \frac{\sqrt{1-8a}}{2} \notin \mathbb{N} \tag{7.111}$$

$$g(y) = w_1(y^2)\, y^{\frac{1-\sqrt{1-8a}}{2}} + w_2(y^2)\, y^{\frac{1+\sqrt{1-8a}}{2}} \ln y, \quad \frac{\sqrt{1-8a}}{2} \in \mathbb{N} \tag{7.112}$$

that is defined and analytic for all complex Δ and $y \neq 0$. Note that a branch cut discontinuity in y is possible.

Proof:

The proof is very similar to that of Corollary 2. Choose a value of k, and set $k(\alpha) = k$. Then the functions $w_1(y)$ and $w_2(y)$ satisfy the growth condition of Theorem 4 for this $k(\alpha)$ and for some $c = c_k$. So by application of Theorem 4, there exists a solution to the partial differential equation (with final condition) $h(\Delta, y)$ that is analytic for all $y \neq 0$ and for all $|\Delta| < k$. Since we can choose any $k > 0$, the circle of analyticity can be shown to be as large as desired. Furthermore, the construction from Theorem 4 does not depend on c or $k(\alpha)$, so it is clear that the functions thus constructed are the same. Consequently, the solution is entire in Δ.

QED

7.11 Proof of Corollary 6

Let $w_1(y)$ and $w_2(y)$ be entire functions, let $c > 0$ be a positive constant, and let $k(\alpha)$ be a positive real continuous function defined on $\alpha \in [0, \pi]$ with $k(0) = k(\pi)$, such that:

$$\left|w_1\left(s^2 e^{2i\alpha}\right)\right| \leq c e^{\frac{s^2}{2k(\alpha)}} \tag{7.113}$$

$$\left|w_2\left(s^2 e^{2i\alpha}\right)\right| \leq c e^{\frac{s^2}{2k(\alpha)}} \tag{7.114}$$

for all real s and $\alpha \in [0, \pi]$. Then there exists a function $h(\Delta, y)$ such that:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) + \left(\frac{a}{y^2} + d\right) h(\Delta, y) \tag{7.115}$$

$$h(0, y) = g(y) \tag{7.116}$$

where:

$$g(y) = w_1(y^2)\, y^{\frac{1-\sqrt{1-8a}}{2}} + w_2(y^2)\, y^{\frac{1+\sqrt{1-8a}}{2}}, \quad \frac{\sqrt{1-8a}}{2} \notin \mathbb{N} \tag{7.117}$$

$$g(y) = w_1(y^2)\, y^{\frac{1-\sqrt{1-8a}}{2}} + w_2(y^2)\, y^{\frac{1+\sqrt{1-8a}}{2}} \ln y, \quad \frac{\sqrt{1-8a}}{2} \in \mathbb{N} \tag{7.118}$$

that is defined and analytic for all complex Δ and $y \neq 0$ such that $\Delta = re^{i\theta}$ for some $\theta \in [0, 2\pi]$ and $0 \leq r < k\left(\frac{\theta}{2}\right)$. Note that a branch cut discontinuity in y is possible.

Proof:

We first consider the problem:

$$\frac{\partial w}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 w}{\partial y^2}(\Delta, y) + \frac{a}{y^2} w(\Delta, y) \tag{7.119}$$

$$w(0, z) = g(y) \tag{7.120}$$

By application of Theorem 4, there exists a solution that is analytic in $y \neq 0$ and in Δ in the region indicated by the condition in the Corollary. We now define:

$$h(\Delta, y) = e^{d\Delta} w(\Delta, y) \tag{7.121}$$

This function is defined for all values of $y \neq 0$, and for all values of Δ where $w(\Delta, y)$ is defined, and is clearly analytic in both variables in this region. Furthermore, the relation is invertible:

$$w(\Delta, y) = e^{-d\Delta} h(\Delta, y) \tag{7.122}$$

Plugging this expression into the PDE specified by Equation 7.119, we find that $h(\Delta, y)$ satisfies the PDE in the Corollary statement. Furthermore, when $\Delta = 0$, from the definition of $w(\Delta, y)$, we have $w(0, y) = h(0, y) = g(y)$, so $h(\Delta, y)$ also satisfies the final condition in the Corollary statement.

QED

7.12 Proof of Corollary 7

Let $w_1(y)$ and $w_2(y)$ be entire functions, let $c > 0$ be a positive constant, let $b \neq 0$ be any non-zero complex number, and let $k(\alpha)$ be a positive real continuous function defined on $\alpha \in [0, \pi]$, such that:

$$\left| e^{-\frac{b}{2}s^2 e^{2i\alpha}} w_1\left(s^2 e^{2i\theta}\right) \right| \leq c e^{\frac{s^2}{2k(\alpha)}} \tag{7.123}$$

$$\left| e^{-\frac{b}{2}s^2 e^{2i\alpha}} w_2\left(s^2 e^{2i\theta}\right) \right| \leq c e^{\frac{s^2}{2k(\alpha)}} \tag{7.124}$$

for all real s and $\alpha \in [0, \pi]$. Then there exists a function $w(\tau, z)$ such that:

$$\frac{\partial w}{\partial \tau}(\tau, z) = \frac{1}{2}\frac{\partial^2 w}{\partial z^2}(\tau, z) - \frac{a}{z^2} w(\tau, z) \tag{7.125}$$

$$w(0, z) = e^{-\frac{\kappa}{2}z^2} g(z) \tag{7.126}$$

where:

$$g(z) = w_1\left(z^2\right) z^{\frac{1-\sqrt{1-8a}}{2}} + w_2\left(z^2\right) z^{\frac{1+\sqrt{1-8a}}{2}}, \quad \frac{\sqrt{1-8a}}{2} \notin \mathbb{N} \tag{7.127}$$

$$g(y) = w_1\left(z^2\right) z^{\frac{1-\sqrt{1-8a}}{2}} + w_2\left(z^2\right) z^{\frac{1+\sqrt{1-8a}}{2}} \ln z, \quad \frac{\sqrt{1-8a}}{2} \in \mathbb{N} \tag{7.128}$$

that is analytic for all complex τ and $z \neq 0$ such that $\tau = re^{i\theta}$ with $\theta \in [0, 2\pi]$ and $0 \leq r < k\left(\frac{\theta}{2}\right)$. Furthermore, the function $h(\Delta, y)$ defined by:

$$h(\Delta, y) = e^{\frac{b}{2}y^2 + \left(\frac{b}{2} - d\right)\Delta} w\left(\frac{e^{2b\Delta} - 1}{2b}, e^{b\Delta} y\right) \tag{7.129}$$

satisfies:

$$\frac{\partial h}{\partial \Delta}(\Delta, y) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(\Delta, y) - \left(\frac{a}{y^2} + \frac{b^2}{2}y^2 + d\right) h(\Delta, y) \tag{7.130}$$

$$h(0, y) = g(y) \tag{7.131}$$

for all complex Δ and y such that:

$$\frac{e^{2b\Delta}-1}{2b}=re^{i\theta} \tag{7.132}$$

for some r and θ, with $\theta \in [0,2\pi]$ and $0 \leq r < k\left(\frac{\theta}{2}\right)$. Note that a branch cut discontinuity in y is possible.

Proof:

We first consider the problem:

$$\frac{\partial w}{\partial \tau}(\tau,z) = \frac{1}{2}\frac{\partial^2 w}{\partial z^2}(\tau,z)-\frac{a}{z^2}w(\tau,z) \tag{7.133}$$

$$w(0,z) = e^{-\frac{b}{2}z^2}g(z) \tag{7.134}$$

By application of Theorem 3, there exists a solution that is analytic in z and in τ in the region indicated by the condition in the Corollary. We now define:

$$\tau(\Delta) = \frac{e^{2b\Delta}-1}{2b} \tag{7.135}$$

$$z(\Delta,y) = e^{b\Delta}y \tag{7.136}$$

$$h(\Delta,y) = e^{\frac{b}{2}y^2+\left(\frac{b}{2}-d\right)\Delta}w(\tau(\Delta),z(\Delta,y)) \tag{7.137}$$

These functions are defined for all values of y, and for all values of Δ such that $\tau(\Delta)$ falls within the indicated region. These equations have a local inverse:

$$\Delta(\tau) = \frac{\ln(1+2b\tau)}{2b} \tag{7.138}$$

$$y(\tau,z) = \frac{z}{\sqrt{1+2b\tau}} \tag{7.139}$$

$$w(\tau,z) = e^{-\frac{b}{2}\left(\frac{z^2}{1+2b\tau}\right)-\left(\frac{b}{2}-d\right)\frac{\ln(1+2b\tau)}{2b}}h(\Delta(\tau),y(\tau,z)) \tag{7.140}$$

Note that it does not matter which logarithm or which square root is chosen in the above equations. Plugging this expression for $w(\tau,z)$ into Equation 7.78, we find that $h(\Delta,y)$ solves the PDE in the Corollary statement. Furthermore, when $\Delta=0$, from the definitions of $\tau(\Delta)$ and $y(\tau,z)$ we have $y(0,z)=z$, and then from the definition of $w(\tau,z)$, we have $w(0,z)=h(0,z)=g(z)$, so $h(\Delta,y)$ also satisfies the final condition in the Corollary statement.

QED