# Bivariate Copula Decomposition in Terms of Comonotonicity, Countermonotonicity and Independence 

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#### Abstract

Copulas are statistical tools for modelling the multivariate dependence structure among variables in a distribution free way. This paper investigates bivariate copula structure, the existence and uniqueness of bivariate copula decomposition in terms of a comonotonic, an independent, a countermonotonic, and an indecomposable part are proved, while the coefficients are determined by partial derivatives of the corresponding copula. Moreover, for the indecomposable part, an optimal convex approximation is provided and analyzed based on the usual criterion. Some applications of the decomposition in finance and insurance are mentioned.

Keywords: Comonotonotic factor, Countermonotonotic factor, Independent factor, Copula decomposition.


## 1 Introduction

A copula is a multivariate distribution function with uniform marginal distributions. It describes the multivariate dependence structure among the random variables in a distribution free way. Sklar's theorem states that for an n-dimensional distribution function $H$ with marginal distributions $F_{1}, \cdots, F_{n}$, there exists an n-copula $C$ such that for all $\left(x_{1}, x_{2}, \cdots, x_{n}\right) \in \bar{R}^{n}$,

$$
\begin{equation*}
H\left(x_{1}, x_{2}, \cdots, x_{n}\right)=C\left(F_{1}\left(x_{1}\right), F_{2}\left(x_{2}\right), \cdots, F_{n}\left(x_{n}\right)\right) \tag{1.1}
\end{equation*}
$$

[^0]If all the marginal distributions are continuous, then $C$ is unique.
In two dimensional case, every copula is bounded by two copulas $C^{+}(u, v)=$ $\min \{u, v\}$ and $C^{-}(u, v)=\max \{u+v-1,0\}$, i.e.,

$$
C^{-}(u, v) \leq C(u, v) \leq C^{+}(u, v)
$$

Copula $C^{-}$is often called the lower Fréchet - Hoeffding bound, and copula $C^{+}$ is called the upper Fréchet - Hoeffding bound. Another well-known copula is the so-called independence copula,

$$
C^{\perp}(u, v)=u v,
$$

which is very common in statistics and probability for modelling sequences of independent experiments.

Sklar's Theorem essentially states that in multivariate setting, marginal distributions and the dependence between observations can be treated separately. This is of great importance for practical work, especially in economics and finance field which always try finding dependence among different random variables. The use of the copula function allows us to overcome the issue of estimating the multivariate distribution function by splitting the distribution function into two unrelated parts:

1. estimate the marginal distributions by fitting the corresponding data via choosing the proper statistical methods;
2. determine the dependence structure of the random variables by specifying a meaningful copula function.

Therefore it has been of great interest to researchers for a long time, see Schweizer (1991), Schweizer and Wolff (1981), Joe (1997), Mari and Kotz (2001) and Nelsen (1991).

On the other hand, in recent years there has been an increasing attention on the dependence structure in finance and insurance. This is due to the fact that the dependence structure is more critical, such as in risk management (for risk measure or pricing), in portfolio selection for hedging or other purposes, in aggregate claims of an insurance portfolio over a certain future reference period, etc. There are many papers dealing with these problems by focusing on copula approach. See Muller and Scarsini (2001), Sarathy, Muralidhar and Parsa (2002), Jondeau and Rockinger (2002), Smith (2003), Alink, Löwe and Wüthrich (2004), Frees and Valdez (1996) and the references therein.

Recently, three important correlations-comonotonicity, countermonotonicity and independence, play significant roles in insurance and finance. The term "comonotonic" comes from 'common monotonic' and is discussed by Schmeidler (1986) and Denneberg (1994). According to Denneberg (1994, pp.54-55), two random variables $X$ and $Y$ are said to be comonotonic, if there exist a random variable $Z$ and two non-decreasing functions $f$ and $g$ such that $X=f(Z)$
and $Y=g(Z)$. Comonotonicity is an extreme positive correlation, it is a deterministic correlation. When $X$ and $Y$ are comonotonic, the outcomes of $X$ and $Y$ always move in the same direction, then neither of them can hedge against the other. The key role of comonotonic can be seen in dual theory (Yaari, 1987), Wang's premium principle (Wang, et al,1997), stop-loss orders and risk measures (Dhaene, et al, 2002a, 2002b). Another extreme case is countermonotonic, an exact opposite of the comonotonic situation, where $X$ and $Y$ are said to be countermonotonic if $X$ and $-Y$ are comonotonic. Countermonotonicity is also important in two dimensional case, see Dhaene, et al (2002b), Embrechets, et al (2001). As we all have known, independence is a very important correlation in describing the dependence structure of risks.

Initiated by the important roles of the above three correlations, for arbitrary two random variables it is interesting to find out their dependence structure in terms of the above dependencies. Thanks for the relationship between the three correlations and the copula functions $C^{+}, C^{-}$and $C^{\perp}$. Theorem 2.5.4 of Nelsen (1999) stated that $X$ and $Y$ are almost surely increasing functions of each other if and only if their joint distribution function equals its Fréchet - Hoeffding upper bounds i.e., $\min \{P(X \leq x), P(Y \leq y)\}$. In another words, the fact that $X$ and $Y$ are comonotonic is equivalent to that their copula equals $C^{+}$. Similarly, that $X$ and $Y$ are countermonotonic is equivalent to that their copula function equals $C^{-}$. Obviously, $X$ and $Y$ are independent if and only if their copula function equals $C^{\perp}$. The above equivalence relationships allow us to focus our discussion on the structure of copulas.

In this paper, for a bivariate copula $C$ we first define its comonotonic factor, its countermonotonic factor and its independent factor, respectively. Then we consider the following decomposition

$$
\begin{equation*}
C(u, v)=\alpha C^{+}(u, v)+\beta C^{\perp}(u, v)+\gamma C^{-}(u, v)+l G(u, v) \tag{1.2}
\end{equation*}
$$

where $\alpha, \beta, \gamma, l \geq 0$ and $G$ is a copula. The first three terms of the above convex sum correspond to the comonotonic part, the independent part and the countermonotonic part, respectively. A copula $C$ is called indecomposable, if for each decomposition of the above form it is necessarily that $\alpha=\beta=\gamma=0$. We will show that a copula $C$ can be decomposed uniquely as a convex combination of a comonotonic part, an independent part, a countermonotonic part, and an indecomposable part, while the first three coefficients of the decomposition are its comonotonic factor, its countermonotonic factor and its independent factor. Moreover, we will also show that the three factors can be determined by partial derivatives of the corresponding copula. For an indecomposable copula, an optimal approximation by the convex combination of comonomtonic part, countermonotonic part and independent part is provided under the usual criterion.

This paper is organized as follows: Section 2 gives our main results on bivariate copula decomposition in terms of comonotonicity, countermonotonicity
and independence, while the method to determine the coefficient of each term in the decomposition is also given. Section 3 presents the convex combination of $C^{+}, C^{-}$and $C^{\perp}$ to approximate the indecomposable part under some criteria. In Section 4 the random mechanic on the convex decomposition is presented. In Section 5 some applications of our results in finance and insurance are briefly provided. Sections 6-7 give the mathematical derivation of our results. Section 8 draws conclusions.

## 2 The convex decomposition of bivariate copula

For a copula $C$, its comonotonic factor $\alpha_{C}$ is defined as the largest $\alpha \in[0,1]$ such that

$$
C(u, v)=\alpha C^{+}(u, v)+(1-\alpha) B(u, v)
$$

here $B(u, v)$ is also a copula. Similarly, its independent factor $\beta_{C}$ and countermonotonic factor are defined as the largest $\beta \in[0,1]$ and the largest $\gamma \in[0,1]$ such that

$$
C(u, v)=\beta C^{\perp}(u, v)+(1-\beta) D(u, v)
$$

and

$$
C(u, v)=\gamma C^{-}(u, v)+(1-\gamma) E(u, v)
$$

here $D$ and $E$ are copulas.
For a bivariate function $g$, we denote

$$
\triangle_{\left(x_{1}, y_{1}\right)}^{\left(x_{2}, y_{2}\right)}(g)=g\left(x_{1}, y_{1}\right)+g\left(x_{2}, y_{2}\right)-g\left(x_{1}, y_{2}\right)-g\left(x_{2}, y_{1}\right)
$$

where $x_{1} \leq x_{2}$ and $y_{1} \leq y_{2}$. It is easy to obtain that

$$
\begin{equation*}
\alpha_{C}=\sup \left\{a: \triangle_{\left(x_{1}, y_{1}\right)}^{\left(x_{2}, y_{2}\right)}(C) \geq a \triangle_{\left(x_{1}, y_{1}\right)}^{\left(x_{2}, y_{2}\right)}\left(C^{+}\right) \text {for all } \quad x_{2} \geq x_{1}, y_{2} \geq y_{1}\right\} \tag{2.1}
\end{equation*}
$$

Let $\mu_{C}$ and $\mu_{C^{+}}$be the probability measures induced by $C$ and $C^{+}$, respectively. Then for $x_{1} \leq x_{2}$ and $y_{1} \leq y_{2}$,
$\mu_{C}\left(\left(x_{1}, y_{1}\right] \times\left(x_{2}, y_{2}\right]\right)=\triangle_{\left(x_{1}, y_{1}\right)}^{\left(x_{2}, y_{2}\right)}(C), \quad \mu_{C^{+}}\left(\left(x_{1}, y_{1}\right] \times\left(x_{2}, y_{2}\right]\right)=\triangle_{\left(x_{1}, y_{1}\right)}^{\left(x_{2}, y_{2}\right)}\left(C^{+}\right)$.
Thus the comonotonic factor can be expressed as

$$
\begin{gathered}
\alpha_{C}=\quad \sup \left\{a: \mu_{C}\left(\left(x_{1}, y_{1}\right] \times\left(x_{2}, y_{2}\right]\right) \geq a \mu_{C^{+}}\left(\left(x_{1}, y_{1}\right] \times\left(x_{2}, y_{2}\right]\right)\right. \\
\text { for all } \left.\quad x_{2} \geq x_{1}, \quad y_{2} \geq y_{1}\right\}
\end{gathered}
$$

From measure theory we know that

$$
\alpha_{C}=\sup \left\{a: \mu_{C}(B) \geq a \mu_{C^{+}}(B) \quad \text { for all Borel set } B \subseteq[0,1] \times[0,1]\right\}
$$

Similarly, the independent factor of $C$ can be expressed as

$$
\begin{equation*}
\beta_{C}=\sup \left\{b: \triangle_{\left(x_{1}, y_{1}\right)}^{\left(x_{2}, y_{2}\right)}(C) \geq b \triangle_{\left(x_{1}, y_{1}\right)}^{\left(x_{2}, y_{2}\right)}\left(C^{\perp}\right) \text { for all } \quad x_{2} \geq x_{1}, y_{2} \geq y_{1}\right\} \tag{2.2}
\end{equation*}
$$

and the countermonotonic factor of $C$ can be expressed as

$$
\begin{equation*}
\gamma_{C}=\sup \left\{c: \triangle_{\left(x_{1}, y_{1}\right)}^{\left(x_{2}, y_{2}\right)}(C) \geq c \triangle_{\left(x_{1}, y_{1}\right)}^{\left(x_{2}\right) y_{2}}\left(C^{-}\right) \text {for all } \quad x_{2} \geq x_{1}, y_{2} \geq y_{1}\right\} . \tag{2.3}
\end{equation*}
$$

Note that the comonotonic factor $\alpha_{C}$ accounts for the portion of positive deterministic relationship between two random variables, the countermonotonic factor $\gamma_{C}$ accounts for the portion of negative deterministic relationship, and the independent factor $\beta_{C}$ gives the portion of independent part.

Here we need some notations. Let

$$
\mathcal{D}_{C, 1}=\left\{(u, v) \in[0,1]^{2}: \frac{\partial}{\partial u} C(u, v) \text { exists }\right\}
$$

and

$$
\mathcal{D}_{C, 2}=\left\{(u, v) \in[0,1]^{2}: \frac{\partial}{\partial v} C(u, v) \text { exists }\right\} .
$$

Denote

$$
\begin{aligned}
& M_{1}(u)=\lim _{v \downarrow u,(u, v) \in \mathcal{D}_{C, 1}} \frac{\partial C(u, v)}{\partial u}-\lim _{v \uparrow u,(u, v) \in \mathcal{D}_{C, 1}} \frac{\partial C(u, v)}{\partial u}, \\
& M_{2}(v)=\lim _{u \downarrow v,(u, v) \in \mathcal{D}_{C, 2}} \frac{\partial C(u, v)}{\partial v}-\lim _{u \uparrow v,(u, v) \in \mathcal{D}_{C, 2}} \frac{\partial C(u, v)}{\partial v}, \\
& M_{3}(u)=\lim _{v \uparrow u,(u, 1-v) \in \mathcal{D}_{C, 1}} \frac{\partial C(u, 1-v)}{\partial u}-\lim _{v \downarrow u,(u, 1-v) \in \mathcal{D}_{C, 1}} \frac{\partial C(u, 1-v)}{\partial u}, \\
& M_{4}(v)=\lim _{u \uparrow v,(1-u, v) \in \mathcal{D}_{C, 2}} \frac{\partial C(1-u, v)}{\partial v}-\lim _{u \downarrow v,(1-u, v) \in \mathcal{D}_{C, 2}} \frac{\partial C(1-u, v)}{\partial v} .
\end{aligned}
$$

Note that $\frac{\partial}{\partial u} C(u, v), \frac{\partial}{\partial v} C(u, v)$ and $\frac{\partial^{2}}{\partial u \partial v} C(u, v)$ exist for almost all $(u, v) \in$ $[0,1]^{2}$ with respect to Lebesgue measure (Theorem 7.1.8 of Lojasiewicz (1988) and Theorem 2.2.7 of Nelsen (1991)). For each fixed $u \in(0,1), M_{1}(u)$ is the size of the jump discontinuity in $\frac{\partial C(u, v)}{\partial u}$ at $v=u$, and $M_{3}(u)$ is the size of the jump discontinuity in $\frac{\partial C(u, v)}{\partial u}$ at $v=1-u$. For each fixed $v \in(0,1), M_{2}(v)$ is the size of the jump discontinuity in $\frac{\partial C(u, v)}{\partial v}$ at $u=v$, and $M_{4}(v)$ is the size of the jump discontinuity in $\frac{\partial C(u, v)}{\partial v}$ at $u=1-v$.

Another notation is also needed. For one measurable function $h(x), x \in R^{2}$, its essential infimum in a measurable set $A \in R^{2}$, is denoted as $g=\operatorname{essinf}_{A} h(x)$.

Theorem 2.1. (1) Each copula $C$ can be decomposed as a convex combination (1.2) of a comonotonic, an independent, a countermonotonic, and an indecomposable part with $\alpha, \beta, \gamma, l \geq 0$. Such a decomposition is unique, and the coefficients $\alpha, \beta, \gamma, l$ equal to the factors $\alpha_{C}, \beta_{C}, \gamma_{C}$ and $l_{C}$, respectively, where $l_{C}=1-\alpha_{C}-\beta_{C}-\gamma_{C}$.
(2) The factors $\alpha_{C}, \beta_{C}$ and $\gamma_{C}$ can be computed by using the following essential infimums,

$$
\begin{align*}
& \beta_{C}=\operatorname{essinf}_{(u, v) \in[0,1]^{2}} \frac{\partial^{2} C(u, v)}{\partial u \partial v},  \tag{2.4}\\
& \alpha_{C}=\operatorname{essinf}_{u \in[0,1]} M_{1}(u)=\operatorname{essinf}_{v \in[0,1]} M_{2}(v),  \tag{2.5}\\
& \gamma_{C}=\operatorname{essinf}_{u \in[0,1]} M_{3}(u)=\operatorname{essinf}_{v \in[0,1]} M_{4}(v), \tag{2.6}
\end{align*}
$$

here every essential infimum in the above equations exists and

$$
\begin{equation*}
0 \leq \alpha_{C}+\beta_{C}+\gamma_{C} \leq 1 \tag{2.7}
\end{equation*}
$$

This theorem presents the relationship between the three factors $\alpha_{C}, \beta_{C}$ and $\gamma_{C}$ and the convex decomposition (1.2). The uniqueness of the decomposition allows us to deal with two random variables' correlation by focusing on its every part separately. This theorem can also be generalized. For two random variables $X$ and $Y$ with copula $C$ and marginal distributions $F_{1}$ and $F_{2}$, applying $C$ 's convex decomposition, the joint distribution function $H$ of $(X, Y)$ can be expressed as

$$
\begin{align*}
H(x, y)= & \alpha_{C} C^{+}\left(F_{1}(x), F_{2}(y)\right)+\beta_{C} C^{\perp}\left(F_{1}(x), F_{2}(y)\right) \\
& +\gamma_{C} C^{-}\left(F_{1}(x), F_{2}(y)\right)+l_{C} G_{C}\left(F_{1}(x), F_{2}(y)\right) . \tag{2.8}
\end{align*}
$$

Example 1: Normal copula is given by

$$
C(u, v)=\Phi_{\rho}\left(\Phi^{-1}(u), \Phi^{-1}(v)\right)
$$

where $\Phi_{\rho}$ is bivariate normal distribution with standard normal marginal distributions and the correlation coefficient $-1<\rho<1, \Phi^{-1}$ is the inverse function of standard normal distribution. One can verify that $\alpha_{C}=\gamma_{C}=0$. In the case $\rho \neq 0$,

$$
\beta_{C}=\operatorname{essinf}_{(u, v) \in[0,1]^{2}} \frac{\partial^{2}}{\partial u \partial v} \Phi_{\rho}\left(\Phi^{-1}(u), \Phi^{-1}(v)\right)=0 .
$$

Thus the normal copula is indecomposable when $\rho \neq 0$.
Example 2: Consider the copula ( Farlie-Gumbel-Morgenstern family)

$$
C_{\theta}(u, v)=u v+\theta u v(1-u)(1-v),
$$

with the parameter $\theta \in[-1,1]$. By detailed calculation one can obtain that

$$
\beta_{C_{\theta}}=1-|\theta|, \quad \alpha_{C_{\theta}}=\gamma_{C_{\theta}}=0
$$

For $\theta \neq 0$, the indecomposable part of $C_{\theta}$ is $C_{1}$ when $\theta>0$ and $C_{-1}$ when $\theta<0$.

Example 3: Consider the copula

$$
C(u, v)=C^{\perp}(u, v)(1-\log \max \{u, v\}), \quad(u, v) \in[0,1]^{2} .
$$

For this copula, $\alpha_{C}=\beta_{C}=\gamma_{C}=0$. It is indecomposable.
Example 4: Consider the two-parameter comprehensive family (Nelsen (1999, p.13) or Kass, et al (2001, p.264))

$$
C(u, v)=a_{1} C^{+}(u, v)+\left(1-a_{1}-a_{2}\right) C^{\perp}(u, v)+a_{2} C^{-}(u, v)
$$

with $a_{1} \geq 0, a_{2} \geq 0$ and $a_{1}+a_{2} \leq 1$. The copula $C$ is the convex combination of comonotonic copula, independent copula and countermonotonic copula. Its comonotonic factor, independent factor and countermonotonic factor are $a_{1}$, $1-a_{1}-a_{2}$ and $a_{2}$ respectively. In the case $a_{1}>0$, its comonotonic factor is greater than zero. In the next section, the comprehensive copula will be used to approximate indecomposable copula.

From the normal copula it can be seen that two random variables may be positive correlated, however their comonotonic factor may equal to zero. Example 2 tells us that for two correlated random variables, some portion of them may be independent. Examples 3 gives another indecomposable copula. Example 4 gives an example that the comonotonic factor is positive.

The next corollary gives the necessary and sufficient condition for a copula to be indecomposable.

Corollary 2.2. Copula $C$ is indecomposable if and only if the following conditions hold:

$$
\operatorname{essinf}_{(u, v) \in[0,1]^{2}} \frac{\partial^{2} C(u, v)}{\partial u \partial v}=\operatorname{essinf}_{u \in[0,1]} M_{1}(u)=\operatorname{essinf}_{u \in[0,1]} M_{3}(u)=0
$$

For the copula $C$ with continuous density function over $[0,1] \times[0,1], \alpha_{C}=$ $\gamma_{C}=0$ and $\beta_{C}$ equals the essential infimum of the joint density function over $[0,1] \times[0,1]$. In this case the possible decomposition of the copula is the sum of the independent part and the indecomposable part. Moreover, $C$ is indecomposable if and only if

$$
\operatorname{essinf}_{(u, v) \in[0,1]^{2}} \frac{\partial^{2} C(u, v)}{\partial u \partial v}=0
$$

## 3 Approximation of the indecomposable part

Usually, for comonotonic, countermonotonic and independent random variables, it is easy to tackle them, such as in Wang's premium principle, (Wang, et al, 1997), dual theory (Yaari, 1987), stop-loss orders and risk measures
(Dhaene, et al, 2002a, 2002b). For the indecomposable part, it benefits practical application to make it sense. In this section, we use the convex sum of $C^{+}, C^{-}$and $C^{\perp}$ to approximate the indecomposable part.

Let $G$ be a copula. Here it can be an indecomposable part of copula $C$, or an arbitrary copula.

Denote the objective function

$$
\begin{aligned}
& s\left(a_{1}, a_{2}\right)=\int_{0}^{1} \int_{0}^{1}[G(u, v) \\
& \left.\quad-\left(a_{1} C^{+}(u, v)+a_{2} C^{-}(u, v)+\left(1-a_{1}-a_{2}\right) C^{\perp}(u, v)\right)\right]^{2} d u d v
\end{aligned}
$$

Our approximation principle is to find out $0 \leq a_{1}^{*} \leq 1, \quad 0 \leq a_{2}^{*} \leq 1$ satisfying $a_{1}^{*}+a_{2}^{*} \leq 1$, such that

$$
\begin{equation*}
s\left(a_{1}^{*}, a_{2}^{*}\right)=\min _{\left\{0 \leq a_{1}, a_{2} \leq 1,\right.} s\left(a_{\left.a_{1}+a_{2} \leq 1\right\}}, a_{2}\right) . \tag{3.1}
\end{equation*}
$$

One interesting fact is that $s\left(a_{1}, a_{2}\right)=0$ if and only if

$$
G(u, v)=a_{1} C^{+}(u, v)+a_{2} C^{-}(u, v)+\left(1-a_{1}-a_{2}\right) C^{\perp}(u, v) .
$$

Theorem 2.1 guarantees that the above decomposition is unique.
It is not easy to solve the above constrained optimization problem directly. By the way, we can overcome the difficulty by solving several unconstrained optimization problems.

First solve the unconstrained optimization problem

$$
\begin{equation*}
s\left(b_{1}^{*}, b_{2}^{*}\right)=\min _{\left\{b_{1}, b_{2}\right\}} s\left(b_{1}, b_{2}\right) \tag{3.2}
\end{equation*}
$$

for the optimal $b_{1}^{*}$ and $b_{2}^{*}$. The detailed derivation (see Section 7) shows that

$$
\begin{align*}
b_{1}^{*}= & 720 \int_{0}^{1} \int_{0}^{1} G(u, v) \\
& \times\left(\frac{8}{15} C^{+}(u, v)+\frac{7}{15} C^{-}(u, v)-C^{\perp}(u, v)\right) d u d v-2  \tag{3.3}\\
b_{2}^{*}= & 720 \int_{0}^{1} \int_{0}^{1} G(u, v) \\
& \times\left(\frac{7}{15} C^{+}(u, v)+\frac{8}{15} C^{-}(u, v)-C^{\perp}(u, v)\right) d u d v \tag{3.4}
\end{align*}
$$

and

$$
\begin{aligned}
s\left(b_{1}^{*}, b_{2}^{*}\right)= & \int_{0}^{1} \int_{0}^{1}(G(u, v))^{2} d u d v-2 \int_{0}^{1} \int_{0}^{1} G(u, v) C^{\perp}(u, v) d u d v \\
& -b_{1}^{*} \int_{0}^{1} \int_{0}^{1} G(u, v)\left(C^{+}(u, v)-C^{\perp}(u, v)\right) d u d v \\
& -b_{2}^{*} \int_{0}^{1} \int_{0}^{1} G(u, v)\left(C^{-}(u, v)-C^{\perp}(u, v)\right) d u d v \\
& +\frac{1}{45} b_{1}^{*}-\frac{7}{360} b_{2}^{*}+\frac{1}{9}
\end{aligned}
$$

If the solution satisfies that $0 \leq b_{1}^{*} \leq 1, \quad 0 \leq b_{2}^{*} \leq 1, \quad 0 \leq 1-b_{1}^{*}-b_{2}^{*} \leq 1$, then the optimal value $s\left(a_{1}^{*}, a_{2}^{*}\right)$ is obtained at points $a_{1}^{*}=b_{1}^{*}, \quad a_{2}^{*}=b_{2}^{*}$. Otherwise, $s\left(a_{1}^{*}, a_{2}^{*}\right)$ must be achieved at the boundary of $[0,1] \times[0,1] \times[0,1]$. So in the following we consider the situation that $a_{1}^{*}=0, a_{2}^{*}=0$ or $a_{1}^{*}+a_{2}^{*}=1$. Let $c_{1}^{*}, c_{2}^{*}, c_{3}^{*}$ be the solutions of the following unconstrained optimization problems

$$
s_{1}\left(c_{1}^{*}\right)=\min _{\left\{c_{1}\right\}} s\left(c_{1}, 0\right), \quad s_{2}\left(c_{2}^{*}\right)=\min _{\left\{c_{2}\right\}} s\left(0, c_{2}\right), \quad s_{3}\left(c_{3}^{*}\right)=\min _{\left\{c_{3}\right\}} s\left(c_{3}, 1-c_{3}\right) .
$$

One can get that

$$
\begin{aligned}
& c_{1}^{*}=90 \int_{0}^{1} \int_{0}^{1} G(u, v)\left(C^{+}(u, v)-C^{\perp}(u, v)\right) d u d v-2 \\
& c_{2}^{*} \quad=90 \int_{0}^{1} \int_{0}^{1} G(u, v)\left(C^{-}(u, v)-C^{\perp}(u, v)\right) d u d v+\frac{7}{4} \\
& c_{3}^{*} \quad=24 \int_{0}^{1} \int_{0}^{1} G(u, v)\left(C^{+}(u, v)-C^{-}(u, v)\right) d u d v-\frac{1}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
s_{1}\left(c_{1}^{*}\right)= & \int_{0}^{1} \int_{0}^{1}(G(u, v))^{2} d u d v-2 \int_{0}^{1} \int_{0}^{1} G(u, v) C^{\perp}(u, v) d u d v+\frac{1}{9} \\
& -c_{1}^{*} \int_{0}^{1} \int_{0}^{1} G(u, v)\left(C^{+}(u, v)-C^{\perp}(u, v)\right) d u d v+\frac{1}{45} c_{1}^{*}, \\
s_{2}\left(c_{2}^{*}\right)= & \int_{0}^{1} \int_{0}^{1}(G(u, v))^{2} d u d v-2 \int_{0}^{1} \int_{0}^{1} G(u, v) C^{\perp}(u, v) d u d v+\frac{1}{9} \\
& -c_{2}^{*} \int_{0}^{1} \int_{0}^{1} G(u, v)\left(C^{-}(u, v)-C^{\perp}(u, v)\right) d u d v-\frac{7}{360} c_{2}^{*}, \\
s_{3}\left(c_{3}^{*}\right) & \int_{0}^{1} \int_{0}^{1}(G(u, v))^{2} d u d v-2 \int_{0}^{1} \int_{0}^{1} G(u, v) C^{-}(u, v) d u d v+\frac{1}{12} \\
& -c_{3}^{*} \int_{0}^{1} \int_{0}^{1} G(u, v)\left(C^{+}(u, v)-C^{-}(u, v)\right) d u d v+\frac{1}{48} c_{3}^{*} .
\end{aligned}
$$

Detailed calculation yields that

$$
c_{1}^{*}=b_{1}^{*}-\frac{7}{8} b_{2}^{*}, \quad c_{2}^{*}=-\frac{7}{8} b_{1}^{*}+b_{2}^{*}, \quad c_{3}^{*}=\frac{1}{2}\left(b_{1}^{*}-b_{2}^{*}\right)+\frac{1}{2} .
$$

Finally, the minimum value of $s\left(a_{1}, a_{2}\right)$ equals

$$
s\left(a_{1}^{*}, a_{2}^{*}\right)=\min \left\{\min _{\left\{i: 0 \leq c_{i}^{*} \leq 1\right\}}\left\{s_{i}\left(c_{i}^{*}\right)\right\}, s(1,0), s(0,1), s(0,0)\right\} .
$$

Note that $a_{1}^{*}$ and $a_{2}^{*}$ are unique.
The following two examples will explain our approximation methodology, with one solved by analytical method and another by numerical method.

Example 5: For the copula in Example 2, $l_{C}=|\theta|$ and

$$
\begin{aligned}
& \int_{0}^{1} \int_{0}^{1} C(u, v) C^{\perp}(u, v) d u d v=\frac{1}{9}+\frac{1}{144} \theta \\
& \int_{0}^{1} \int_{0}^{1} C(u, v) C^{+}(u, v) d u d v=\frac{2}{15}+\frac{13}{1260} \theta \\
& \int_{0}^{1} \int_{0}^{1} C(u, v) C^{-}(u, v) d u d v=\frac{11}{120}+\frac{1}{280} \theta, \\
& \int_{0}^{1} \int_{0}^{1} C^{2}(u, v) d u d v=\frac{1}{9}+\frac{1}{72} \theta+\frac{1}{900} \theta^{2}
\end{aligned}
$$

Then

$$
b_{1}^{*}=\frac{17}{105} \theta, \quad b_{2}^{*}=-\frac{17}{105} \theta .
$$

Note that except for the case $\theta=0$, the constraint conditions are not satisfied. Now let $\theta \neq 0$, one can get that

$$
c_{1}^{*}=\frac{17}{56} \theta, \quad c_{2}^{*}=-\frac{17}{56} \theta, \quad c_{3}^{*}=\frac{1}{2}+\frac{17}{105} \theta
$$

and

$$
s\left(c_{1}^{*}, 0\right)=\frac{41}{470400} \theta^{2}, \quad s\left(0, c_{2}^{*}\right)=\frac{41}{470400} \theta^{2}, \quad s\left(c_{3}^{*}, 1-c_{3}^{*}\right)=\frac{1}{1440}+\frac{1}{52920} \theta^{2} .
$$

Since $s\left(c_{3}^{*}, 1-c_{3}^{*}\right)>s\left(c_{1}^{*}, 0\right)=s\left(0, c_{2}^{*}\right)$ for $\theta \in[-1,1]$, we can find the optimal parameters

$$
\left(a_{1}^{*}, a_{2}^{*}\right)= \begin{cases}\left(\frac{17}{56} \theta, 0\right), & 0<\theta \leq 1 \\ \left(-\frac{17}{56} \theta, 0\right), & -1 \leq \theta<0\end{cases}
$$

and

$$
s\left(a_{1}^{*}, a_{2}^{*}\right)=\frac{41}{470400} \theta^{2} .
$$

Note that the approximation error is quite small.

Example 6: For the normal copula in Example 1, it is difficult to get the accurate values of $a_{1}^{*}, a_{2}^{*}$ by analytical method. Using the numerical method, the accurate calculation will be involved in complex approximation to the integrals. Here we will not focus our attention on the accurate calculating. For illustrate purpose we list some figures by numerical approximation to $b_{1}^{*}, b_{2}^{*}$, etc. Figure 1 shows the trend of $b_{1}^{*}, b_{2}^{*}$. From this graph we can see that the factors $b_{1}^{*}, b_{2}^{*}$ and $1-b_{1}^{*}-b_{2}^{*}$ do not satisfy the constraints. So we should use two terms to approximate the normal copula with respect to different correlation coefficient $\rho$. For negative $\rho$, intuitively, the convex sum of the countermonotonic part and independent part may produce good approximation, the correspondent results are shown in Figure 2. While for positive $\rho$, intuitively, the convex combination of comonotonic part and independent part is more suitable, the results are shown in Figure 3.


Figure 1: Three different factors and the correlation coefficient $\rho$

It is difficult to find upper bound for the constrained optimization problem $s\left(a_{1}^{*}, a_{2}^{*}\right)$ in (3.2). In the following we give an upper bound for the unconstrained optimization problem $\min _{\left\{a_{1}, a_{2}\right\}} s\left(a_{1}, a_{2}\right)$. Recall that in the case $0 \leq b_{1}^{*} \leq$ $1,0 \leq b_{2}^{*} \leq 1$ and $0 \leq 1-b_{1}^{*}-b_{2}^{*} \leq 1, \min _{\left\{a_{1}, a_{2}\right\}} s\left(a_{1}, a_{2}\right)=s\left(a_{1}^{*}, a_{2}^{*}\right)$ follows.


Figure 2: Countermonotonic factor, independent factor and the correlation coefficient $\rho$ when $\rho$ is negative

Theorem 3.1. For arbitrary copula $G$,

$$
\min _{\left\{a_{1}, a_{2}\right\}} s\left(a_{1}, a_{2}\right) \leq \int_{0}^{1} \int_{0}^{1}(G(u, v))^{2} d u d v-\frac{1}{12} .
$$

The proof will be given in Section 7 .

## 4 The probability models on the convex decomposition

Let us introduce uniform $[0,1]$ distributed random variables $U, V$ and $W$ in probability space $(\Omega, \mathcal{F}, P)$ and the disjoint sets $A^{+}, A^{\perp}, A^{-}, A^{I} \in \mathcal{F}$. It is also assumed that the sets $A^{+}, A^{\perp}, A^{-}, A^{I}$ are independent of the random variables $U, V$ and $W$. Suppose that $U$ and $V$ are independent, $U$ and $W$ has joint distribution

$$
G_{C}(u, v)= \begin{cases}\frac{C(u, v)-\alpha_{C} C^{+}(u, v)-\beta_{C} C^{\perp}(u, v)-\gamma_{C} C^{-}(u, v)}{1-\alpha_{C}-\beta_{C}-\gamma_{C}}, & \alpha_{C}+\beta_{C}+\gamma_{C}<1 \\ \Phi_{0.5}\left(\Phi^{\leftarrow}(u), \Phi^{\leftarrow}(v)\right), & \alpha_{C}+\beta_{C}+\gamma_{C}=1\end{cases}
$$

and

$$
P\left(A^{+}\right)=\alpha_{C}, \quad P\left(A^{\perp}\right)=\beta_{C}, \quad P\left(A^{-}\right)=\gamma_{C}, \quad P\left(A^{I}\right)=l_{C} .
$$



Figure 3: Comonotonic factor, independent factor and the correlation coefficient $\rho$ when $\rho$ is positive

Proposition 4.1. Random vector $(\zeta, \eta)$ defined by

$$
\begin{equation*}
(\zeta, \eta)=\left(U, U I_{A^{+}}+V I_{A^{\perp}}+(1-U) I_{A^{-}}+W I_{A^{I}}\right) \tag{4.1}
\end{equation*}
$$

has joint distribution function C. Furthermore,

$$
\begin{aligned}
& P\left(\zeta \leq u, \eta \leq v \mid A^{+}\right)=C^{+}(u, v), \quad P\left(\zeta \leq u, \eta \leq v \mid A^{\perp}\right)=C^{\perp}(u, v), \\
& P\left(\zeta \leq u, \eta \leq v \mid A^{-}\right)=C^{-}(u, v)
\end{aligned}
$$

and if $l_{C} \neq 0$,

$$
P\left(\zeta \leq u, \eta \leq v \mid A^{I}\right)=G_{C}(u, v)
$$

Proof. As a consequence of Mikusinski, Sherwood and Taylor (1991)' result, $(\zeta, \eta)$ has distribution $C$. Moreover,

$$
P\left(\zeta \leq u, \eta \leq v \mid A^{+}\right)=P(U \leq u, U \leq v)=C^{+}(u, v) .
$$

The other equations can be proved similarly.
In fact, $\Phi_{0.5}\left(\Phi^{\leftarrow}(u), \Phi^{\leftarrow}(v)\right)$ can be replaced by an arbitrary indecomposable copula.

The probability space $\Omega$ can be partitioned into four subspaces via the dependent structure of copula $C$. The random variables $\zeta$ and $\eta$ in (4.1) are comonotonic in the subspace $A^{+}$, independent in the subspace $A^{\perp}$ and countermonotonic in the subspace $A^{-}$.

Generally, consider two random variables' dependence. For two random variables $X$ and $Y$ with copula $C$ and marginal distributions $F_{1}$ and $F_{2}$, their joint distribution is given in (2.7). Define the inverse functions of $F_{1}$ and $F_{2}$ respectively by

$$
F_{1}^{\leftarrow}(x)=\inf \left\{s: F_{1}(s) \geq x\right\}, \quad F_{2}^{\leftarrow}(x)=\inf \left\{s: F_{2}(s) \geq x\right\}
$$

Denote

$$
\left(X^{+}, Y^{+}\right)=\left(F_{1}^{\leftarrow}(U), F_{2}^{\leftarrow}(U)\right), \quad\left(X^{\perp}, Y^{\perp}\right)=\left(F_{1}^{\leftarrow}(U), F_{2}^{\leftarrow}(V)\right)
$$

and

$$
\left(X^{-}, Y^{-}\right)=\left(F_{1}^{\leftarrow}(U), F_{2}^{\leftarrow}(1-U)\right), \quad\left(X^{I}, Y^{I}\right)=\left(F_{1}^{\leftarrow}(U), F_{2}^{\leftarrow}(W)\right)
$$

where $U, V$ and $W$ are defined before. The four pairs $\left(X^{+}, Y^{+}\right),\left(X^{\perp}, Y^{\perp}\right)$, $\left(X^{-}, Y^{-}\right)$and $\left(X^{I}, Y^{I}\right)$ have the same marginal distributions as $(X, Y)$ 's. The first pair $\left(X^{+}, Y^{+}\right)$is comonotonic, the second to fourth pairs are independent, countermonotonic and indecomposable, respectively. It is easy to get the following proposition. The proof is omitted here.

Proposition 4.2. For any non-negative measurable function $f$ on $R^{2}$, one has

$$
\begin{aligned}
E[f(X, Y)]= & \alpha_{C} E\left[f\left(X^{+}, Y^{+}\right)\right]+\beta_{C} E\left[f\left(X^{\perp}, Y^{\perp}\right)\right] \\
& +\gamma_{C} E\left[f\left(X^{-}, Y^{-}\right)\right]+l_{C} E\left[f\left(X^{I}, Y^{I}\right)\right]
\end{aligned}
$$

The coefficients of the above decomposition only depend on the copula of $(X, Y)$. This property will show its advantage when applying it to deal with random vectors with the same copula function.

## 5 Applications in insurance and finance

Recently, copula is becoming a hot topic in finance and insurance. This section will give some ideas briefly on applications of our results in these fields.

For a given copula, Theorem 2.1 can be used to get the convex decomposition, four factors can be found. If the indecomposable factor is not small enough, then the approximation method in Section 3 can be used. Hence the four factors and the two values of $a_{1}^{*}, a_{2}^{*}$ (defined in section 3) are obtained. The following comments on its application in insurance and finance aspects are given:

1. In variance's decomposition: From Proposition 4.2 one can verify that for every $a, b \in R$, it holds that

$$
\begin{aligned}
\operatorname{Var}(a X+b Y)= & \alpha_{C} \operatorname{Var}\left(a X^{+}+b Y^{+}\right)+\beta_{C} \operatorname{Var}\left(a X^{\perp}+b Y^{\perp}\right) \\
& +\gamma_{C} \operatorname{Var}\left(a X^{-}+b Y^{-}\right)+l_{C} \operatorname{Var}\left(a X^{I}+b Y^{I}\right)
\end{aligned}
$$

The above equation can be applied to find mean-variance optimal investment portfolio in finance.
2. In stop-loss premium's decomposition: For one risk $Y$, its stop-loss premium is defined as $E(Y-t)_{+}=E(\max \{Y-t, 0\})$, where $t \in[0, \infty)$. The stop-loss premium is applied when ordering risks in insurance (see chapter 10 of Kass, et al. (2001)). From Proposition 4.2 one can find that

$$
\begin{aligned}
E(X+Y-t)_{+} & =\alpha_{C} E\left(X^{+}+Y^{+}-t\right)_{+}+\beta_{C} E\left(X^{\perp}+Y^{\perp}-t\right)_{+} \\
& +\gamma_{C} E\left(X^{-}+Y^{-}-t\right)_{+}+l_{C} E\left(X^{I}+Y^{I}-t\right)_{+}
\end{aligned}
$$

By the well-known fact (Dhaene, et al., 2002a, 2002b)

$$
E\left(X^{-}+Y^{-}-t\right)_{+} \leq E\left(X^{I}+Y^{I}-t\right)_{+} \leq E\left(X^{+}+Y^{+}-t\right)_{+}
$$

we get

$$
\begin{aligned}
E(X+Y-t)_{+} & \leq\left(\alpha_{C}+l_{C}\right) E\left(X^{+}+Y^{+}-t\right)_{+}+\beta_{C} E\left(X^{\perp}+Y^{\perp}-t\right)_{+} \\
& +\gamma_{C} E\left(X^{-}+Y^{-}-t\right)_{+}
\end{aligned}
$$

and

$$
\begin{aligned}
E(X+Y-t)_{+} & \geq \alpha_{C} E\left(X^{+}+Y^{+}-t\right)_{+}+\beta_{C} E\left(X^{\perp}+Y^{\perp}-t\right)_{+} \\
& +\left(\gamma_{C}+l_{C}\right) E\left(X^{-}+Y^{-}-t\right)_{+}
\end{aligned}
$$

When $l_{C}$ is small enough, the above inequalities provide good approximation.
3. In finance for hedging: Suppose that a company A with risk $X$ wants to find an asset from $\left(Y_{1}, Y_{2}, \cdots, Y_{n}\right)$ existing in the financial market to hedge its risk $X$. Here the factors and values corresponding to $\left(X, Y_{i}\right)$ are denoted by $\alpha_{i}, \beta_{i}, \gamma_{i}, a_{1}^{i}$ and $a_{2}^{i}$. It should choose the one which $\mid \gamma_{i}-$ $\alpha_{i} \mid$ is the largest when $l_{i}$ is small enough, otherwise the larger value of $\left|\left(\gamma_{i}-\alpha_{i}\right)+l_{i}\left(a_{2}^{i}-a_{1}^{i}\right)\right|$, the better for hedging.

## 6 Proof of Theorem 2.1

Let $\mathcal{S C}=\left\{x_{1}, x_{2}, \cdots\right\}$ be a countable dense set of $[0,1]$, and denote

$$
\mathcal{A}_{i}=\left\{u \in[0,1]: \frac{\partial}{\partial u} C\left(u, x_{i}\right) \text { exists }\right\}
$$

and $\mathcal{A}_{C}=\cap_{i=1} \mathcal{A}_{i}$. For fixed $i, C\left(u, x_{i}\right)$ is increasing about $u \in[0,1]$, then $\frac{\partial}{\partial u} C\left(u, x_{i}\right)$ exists for almost all $u \in[0,1]$. Thus the lebesgue measure of the set $[0,1] / \mathcal{A}_{C}$ is equal to zero. Further, $\frac{\partial}{\partial u} C(u, v)$ exists for $(u, v) \in \mathcal{A}_{C} \times \mathcal{S C}$, and $\frac{\partial}{\partial u} C(u, v)$ is increasing with respect to $v$. See Nelsen (1991, Part II of Theorem 2.2.7, p11).

Lemma 6.1. For almost all $u \in[0,1], M_{i}(u)$ exists and is non-negative.
Proof. Here we only prove the case $i=1$. It suffices to prove that $M_{1}(u)$ exists for $u \in \mathcal{A}_{C}$.

Fix $u \in \mathcal{A}_{C}$. When $v$ is in the dense set $\mathcal{S C}, \frac{\partial}{\partial u} C(u, v)$ exists and $\frac{\partial}{\partial u} C(u, v)$ is increasing about $v$, hence $M_{1}(u)$ exists for $u \in \mathcal{A}_{C}$ and $M_{1}(u) \geq 0$. The lemma is proved.

By Lemma 6.1, $\operatorname{essinf}_{u \in[0,1]} M_{1}(u)$ exists.
Lemma 6.2. Each copula function $C$ can be decomposed as

$$
\begin{equation*}
C(u, v)=\operatorname{essinf}_{u \in[0,1]} M_{1}(u) C^{+}(u, v)+\left(1-\operatorname{essinf}_{u \in[0,1]} M_{1}(u)\right) D(u, v) \tag{6.1}
\end{equation*}
$$

where $D$ is a copula.
Proof. Denote $a=\operatorname{essinf}_{u \in[0,1]} M_{1}(u)$ and

$$
d(u, v)= \begin{cases}\frac{\partial C(u, v)}{\partial u}-a I_{\{v>u\}}, & \text { if } \frac{\partial C(u, v)}{\partial u} \text { exists } \\ 0, & \text { otherwise. }\end{cases}
$$

Fix $v \in[0,1]$. By Radon-Nikodym theorem (Chow and Teicher (1988), p195), the copula function $C$ can be decomposed as

$$
\begin{equation*}
C(u, v)=\int_{0}^{u} \frac{\partial C(s, v)}{\partial s} d s+A(u, v) \tag{6.2}
\end{equation*}
$$

where the non-negative function $A(u, v)$ is increasing with respect to $u$. Note that (6.2) can be rearranged as

$$
\begin{equation*}
C(u, v)=f(u, v)+A(u, v)+a C^{+}(u, v) \tag{6.3}
\end{equation*}
$$

where $f(u, v)=\int_{0}^{u} d(s, v) d s$.

In the following, suppose that for all $u_{1} \leq u_{2}, v_{1} \leq v_{2}$

$$
\begin{equation*}
\triangle_{\left(u_{1}, v_{1}\right)}^{\left(u_{2}, v_{2}\right)}(f) \geq 0, \quad \triangle_{\left(u_{1}, v_{1}\right)}^{\left(u_{2}, v_{2}\right)}(A) \geq 0 \tag{6.4}
\end{equation*}
$$

Then $f(u, v)$ and $A(u, v)$ can induce two measures. Since $f(0,0)=A(0,0)=0$, then $f$ and $g$ are non-negative.

Consider the case $a=1$. Letting $u=1, v=1$ in (6.3), $f(1,1)+A(1,1)+1=$ 1, thus $f(1,1)=A(1,1)=0$ follows. Hence $f(u, v)=A(u, v)=0$ and (6.1) follows from (6.3).

Now consider the case $a<1$. In this case, (6.3) can be rewritten as

$$
\begin{equation*}
C(u, v)=(1-a) D(u, v)+a C^{+}(u, v) \tag{6.5}
\end{equation*}
$$

where $D(u, v)=\frac{f(u, v)+A(u, v)}{1-a}$. It is easy to verify that $D(u, v)$ is a copula. Thus (6.1) holds.

To finish the proof of the lemma, it suffices to prove that (6.4) holds. In the following, we will prove that $\triangle_{\left(u_{1}, v_{1}\right)}^{\left(u_{2}, v_{2}\right)}(f) \geq 0$ in (a) and $\triangle_{\left(u_{1}, v_{1}\right)}^{\left(u_{2}, v_{2}\right)}(A) \geq 0$ in (b). For fixed $v_{1}<v_{2}$, denote

$$
\mathcal{T}=\left\{s \in[0,1]:\left(s, v_{1}\right) \in \mathcal{D}_{C, 1}, \quad\left(s, v_{2}\right) \in \mathcal{D}_{C, 1}\right\}
$$

(a) By the monotonicity of $\frac{\partial C(s, v)}{\partial s}$ with respect to $v$ and the definition of $a$, we know that for almost all $s \in \mathcal{T}$,

$$
\begin{aligned}
& \frac{\partial C\left(s, v_{2}\right)}{\partial s}-\frac{\partial C\left(s, v_{1}\right)}{\partial s}-a I_{\left\{v_{2}>s \geq v_{1}\right\}} \\
& = \begin{cases}\frac{\partial C\left(s, v_{2}\right)}{\partial s}-\frac{\partial C\left(s, v_{1}\right)}{\partial s} \geq 0, & s \geq v_{2} \text { or } s<v_{1} \\
\frac{\partial C\left(s, v_{2}\right)}{\partial s}-\frac{\partial C\left(s, v_{1}\right)}{\partial s}-a \geq 0, & v_{1} \leq s<v_{2} .\end{cases}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\triangle_{\left(u_{1}, v_{1}\right)}^{\left(u_{2}, v_{2}\right)}(f) & =\int_{s \in \mathcal{T}, u_{1}<s \leq u_{2}}\left\{\frac{\partial C\left(s, v_{2}\right)}{\partial s}-a I_{\left\{v_{2}>s\right\}}-\frac{\partial C\left(s, v_{1}\right)}{\partial s}+a I_{\left\{v_{1}>s\right\}}\right\} d s \\
& =\int_{s \in \mathcal{T}, u_{1}<s \leq u_{2}}\left\{\frac{\partial C\left(s, v_{2}\right)}{\partial s}-\frac{\partial C\left(s, v_{1}\right)}{\partial s}-a I_{\left\{v_{2}>s \geq v_{1}\right\}}\right\} d s \geq 0
\end{aligned}
$$

(b)From (6.2) we have

$$
\begin{aligned}
\triangle_{\left(u_{1}, v_{1}\right)}^{\left(u_{2}, v_{2}\right)}(C) & =\int_{s \in \mathcal{T}, u_{1}<s \leq u_{2}}\left(\frac{\partial C\left(s, v_{2}\right)}{\partial s}-\frac{\partial C\left(s, v_{1}\right)}{\partial s}\right) d s+\triangle_{\left(u_{1}, v_{1}\right)}^{\left(u_{2}, v_{2}\right)}(A) \\
& =\int_{s \in \mathcal{T}, u_{1}<s \leq u_{2}} \frac{\partial P\left(\zeta \leq s, v_{1}<\eta \leq v_{2}\right)}{\partial s} d s+\triangle_{\left(u_{1}, v_{1}\right)}^{\left(u_{2}, v_{2}\right)}(A),
\end{aligned}
$$

where the random vector $(\zeta, \eta)$ is defined in (4.1) with distribution function $C$. By Canonical Lebesgue Decomposition (Theorem 4.4.9 of Lojasiewicz (1998)),

$$
\triangle_{\left(u_{1}, v_{1}\right)}^{\left(u_{2}, v_{2}\right)}(C)=P\left(u_{1}<\zeta \leq u_{2}, v_{1}<\eta \leq v_{2}\right) \geq \int_{u_{1}}^{u_{2}} \frac{\partial P\left(\zeta \leq s, v_{1}<\eta \leq v_{2}\right)}{\partial s} d s
$$

Thus we conclude that

$$
\triangle_{\left(u_{1}, v_{1}\right)}^{\left(u_{2}, v_{2}\right)}(A) \geq 0
$$

Now we finish the proof of the lemma.

Lemma 6.3. Copula $C$ can be decompensated as

$$
\begin{align*}
C(u, v)= & \operatorname{essinf}_{(u, v) \in[0,1]^{2}} \frac{\partial^{2} C(u, v)}{\partial u \partial v} \times C^{\perp}(u, v) \\
& +\left(1-\operatorname{essinf}_{(u, v) \in[0,1]^{2}} \frac{\partial^{2} C(u, v)}{\partial u \partial v}\right) \times P(u, v), \tag{6.6}
\end{align*}
$$

where $P$ is a copula.
Proof. Denote $b=\operatorname{essinf}_{(u, v) \in[0,1]^{2}} \frac{\partial^{2} C(u, v)}{\partial u \partial v}$. Applying Radon-Nikodym theorem (Chow and Teicher (1988), p195) again, there exists bivariate continuous Lebesgue-singular measure $C_{d}(u, v)$ such that

$$
\begin{equation*}
C(u, v)=\int_{0}^{u} \int_{0}^{v} \frac{\partial^{2}}{\partial s \partial t} C(s, t) d t d s+C_{d}(u, v) \tag{6.7}
\end{equation*}
$$

holds. In the case $b=1, C(u, v)=C^{\perp}(u, v)$ follows. Otherwise (6.7) can be expressed as

$$
\begin{equation*}
C(u, v)=(1-b) P(u, v)+b C^{\perp}(u, v) \tag{6.8}
\end{equation*}
$$

where

$$
P(u, v)=\frac{\int_{0}^{u} \int_{0}^{v}\left(\frac{\partial^{2}}{\partial s \partial t} C(s, t)-b\right) d t d s+C_{d}(u, v)}{1-b}
$$

is a copula. The lemma is proved.

It is easy to proof the next lemma. Here we omit its proof.
Lemma 6.4. Let

$$
L(u, v)=u-C(u, 1-v)
$$

Then $L$ is a copula, and the comonotonic factor $\alpha_{L}$ of $L$ equals $\gamma_{C}$.
Based on the above lemmas, we can prove Theorem 2.1 now.

## Proof of Theorem 2.1

We first prove that (2.4)-(2.6) hold, then prove (2.7) and the first part of the theorem.
(1) Proof of (2.4): According to the definition of $\beta_{C}$, the function $C$ can be decomposed as

$$
C(u, v)=\beta_{C} C^{\perp}(u, v)+\left(1-\beta_{C}\right) G(u, v)
$$

where $G$ is a copula. Then for almost all $(u, v) \in[0,1]^{2}$,

$$
\frac{\partial^{2}}{\partial u \partial v} C(u, v)=\beta_{C}+\left(1-\beta_{C}\right) \frac{\partial^{2}}{\partial u \partial v} G(u, v)
$$

Thus

$$
\operatorname{essinf}_{(u, v) \in[0,1]^{2}}\left\{\frac{\partial^{2}}{\partial u \partial v} C(u, v)\right\} \geq \beta_{C}
$$

On the other hand, from Lemma 6.3 and the definition of $\beta_{C}$ we know that

$$
\beta_{C} \geq \operatorname{essinf}_{(u, v) \in[0,1]^{2}}\left\{\frac{\partial^{2}}{\partial u \partial v} C(u, v)\right\}
$$

Then (2.4) follows.
(2) Proof of (2.5): Note that the copula function $C$ can be decomposed as

$$
C(u, v)=\alpha_{C} C^{+}(u, v)+\left(1-\alpha_{C}\right) R(u, v)
$$

where $R$ is a copula. For $u \neq v$, differentiating the above equation with respect to $u, \frac{\partial R(u, v)}{\partial u}$ exists if and only if $\frac{\partial C(u, v)}{\partial u}$ exists, and the following equation holds for almost all $(u, v) \in[0,1]^{2}$,

$$
\frac{\partial C(u, v)}{\partial u}= \begin{cases}\alpha_{C}+\left(1-\alpha_{C}\right) \frac{\partial R(u, v)}{\partial u}, & u<v \\ \left(1-\alpha_{C}\right) \frac{\partial R(u, v)}{\partial u}, & u>v\end{cases}
$$

Then for every $u \in \mathcal{A}_{C}$, we have

$$
\begin{aligned}
M_{1}(u) & =\alpha_{C}+\left(1-\alpha_{C}\right)\left(\lim _{v \downarrow u,(u, v) \in \mathcal{D}_{C, 1}} \frac{\partial R(u, v)}{\partial u}-\lim _{v \uparrow u,(u, v) \in \mathcal{D}_{C, 1}} \frac{\partial R(u, v)}{\partial u}\right) \\
& \geq \alpha_{C}
\end{aligned}
$$

On the other hand, from the definition of $\alpha_{C}$ and Lemma 6.2 we have

$$
\operatorname{essinf}_{u \in[0,1]} M_{1}(u) \leq \alpha_{C}
$$

Thus

$$
\operatorname{essinf}_{u \in[0,1]} M_{1}(u)=\alpha_{C}
$$

Similarly the other part of (2.5) can be proved.
(3) Proof of (2.6): Let $L$ be defined as in Lemma 6.4. Then

$$
\mathcal{D}_{L, 1}=\left\{(u, v) \in[0,1]^{2}: \frac{\partial}{\partial u} L(u, v) \text { exists }\right\}=\left\{(u, v) \mid(u, 1-v) \in \mathcal{D}_{C, 1}\right\}
$$

and for $(u, v) \in \mathcal{D}_{L, 1}$,

$$
\frac{\partial L(u, v)}{\partial u}=1-\frac{\partial C(u, 1-v)}{\partial u}
$$

holds. From the above equation and (2.5), it yields that

$$
\begin{aligned}
\alpha_{L} & =\operatorname{essinf}_{u \in[0,1]}\left\{\lim _{v \downarrow u,(u, v) \in \mathcal{D}_{L, 1}} \frac{\partial L(u, v)}{\partial u}-\lim _{v \uparrow u,(u, v) \in \mathcal{D}_{L, 1}} \frac{\partial L(u, v)}{\partial u}\right\} \\
& =\operatorname{essinf}_{u \in[0,1]}\left\{\lim _{v \uparrow u,(u, 1-v) \in \mathcal{D}_{C, 1}} \frac{\partial C(u, 1-v)}{\partial u}-\lim _{v \downarrow u,(u, 1-v) \in \mathcal{D}_{C, 1}} \frac{\partial C(u, 1-v)}{\partial u}\right\} .
\end{aligned}
$$

Hence the first part of (2.6) can be obtained by using Lemma 6.4. Similarly the second part can be obtained.
(4) We will show that (1.2) holds with

$$
\alpha=\alpha_{C}, \beta=\beta_{C}, \gamma=\gamma_{C}, l_{C}=1-\alpha_{C}-\beta_{C}-\gamma_{C}
$$

and that (2.7) holds.
Based on the definition of $\alpha_{C}$,

$$
C(u, v)=\alpha_{C} C^{+}(u, v)+\left(1-\alpha_{C}\right) D_{1}(u, v)
$$

Note that $1-\alpha_{C} \geq 0$ and $D_{1}$ is a copula. For the copula $D_{1}$ it holds that

$$
D_{1}(u, v)=\beta_{D_{1}} C^{\perp}(u, v)+\left(1-\beta_{D_{1}}\right) D_{2}(u, v)
$$

here $D_{2}$ is a copula. Thus

$$
C(u, v)=\alpha_{C} C^{+}(u, v)+\left(1-\alpha_{C}\right) \beta_{D_{1}} C^{\perp}(u, v)+\left(1-\alpha_{C}\right)\left(1-\beta_{D_{1}}\right) D_{2}(u, v)
$$

follows. By (2.4) and the above decomposition one can verify that $\beta_{C}=$ $\left(1-\alpha_{C}\right) \beta_{D_{1}}$. Similarly, for

$$
D_{2}(u, v)=\gamma_{D_{2}} C^{-}(u, v)+\left(1-\gamma_{D_{2}}\right) G(u, v)
$$

we have

$$
\begin{aligned}
C(u, v)= & \alpha_{C} C^{+}(u, v)+\beta_{C} C^{\perp}(u, v)+\left(1-\alpha_{C}\right)\left(1-\beta_{D_{1}}\right) \gamma_{D_{2}} C^{-}(u, v) \\
& +\left(1-\alpha_{C}\right)\left(1-\beta_{D_{1}}\right)\left(1-\gamma_{D_{2}}\right) G(u, v)
\end{aligned}
$$

and $\gamma_{C}=\left(1-\alpha_{C}\right)\left(1-\beta_{D_{1}}\right) \gamma_{D_{2}}$. Moreover,

$$
1-\alpha_{C}-\beta_{C}-\gamma_{C}=\left(1-\alpha_{C}\right)\left(1-\beta_{D_{1}}\right)\left(1-\gamma_{D_{2}}\right) \geq 0
$$

Thus (1.2) holds with $\alpha=\alpha_{C}, \beta=\beta_{C}, \gamma=\gamma_{C}, l_{C}=1-\alpha_{C}-\beta_{C}-\gamma_{C}$ and (2.7) follows.
(5) Proof of the first part of Theorem 2.1: Suppose that (1.2) holds for the two cases $\alpha=\alpha_{1}, \beta=\beta_{1}, \gamma=\gamma_{1}, G=G_{1}$ and $\alpha=\alpha_{2}, \beta=\beta_{2}, \gamma=\gamma_{2}, G=G_{2}$ respectively, with $l_{1}, l_{2} \geq 0$. Further, it is also assumed that $G_{1}$ is indecomposable if $l_{1} \neq 0$, and $G_{2}$ is indecomposable if $l_{2} \neq 0$.
(a) First we will prove that $\beta_{1}=\beta_{2}$.

The case $\beta_{1}=1$ or $\beta_{2}=1$ is trivial. In the next we consider the case that $\beta_{1}<1, \beta_{2}<1$. Notice that

$$
\begin{align*}
& \alpha_{1} C^{+}(u, v)+\gamma_{1} C^{-}(u, v)+l_{1} G_{1}(u, v) \\
& =\alpha_{2} C^{+}(u, v)+\gamma_{2} C^{-}(u, v)+l_{2} G_{2}(u, v)+\left(\beta_{2}-\beta_{1}\right) C^{\perp}(u, v) \tag{6.9}
\end{align*}
$$

Suppose $1>\beta_{2}>\beta_{1}$ and let

$$
P(u, v)=\frac{\alpha_{1} C^{+}(u, v)+\gamma_{1} C^{-}(u, v)+l_{1} G_{1}(u, v)}{1-\beta_{1}}
$$

Then

$$
\frac{\partial^{2} P(u, v)}{\partial u \partial v}=\frac{l_{1}}{1-\beta_{1}} \frac{\partial^{2} G_{1}(u, v)}{\partial u \partial v}, \quad \text { a.e.. }
$$

Using (6.9) and the fact that $\beta_{2}-\beta_{1}>0$, the independent factor $\beta_{P}$ of $P(u, v)$ is greater than zero. Then by (2.4) and the above equation we assert that $l_{1}>0$ and the independent factor of $G_{1}$ is greater than zero, which contradicts to the indecomposable assumption on $G_{1}$. Thus $\beta_{2} \leq \beta_{1}$ follows. Similarly, we can get $\beta_{1} \leq \beta_{2}$ and finally $\beta_{1}=\beta_{2}$ holds.
(b) Since $\beta_{1}=\beta_{2}$ holds, for simplicity we only consider the case $\beta_{1}=\beta_{2}=$ 0 . In the following we give the proof of $\alpha_{1}=\alpha_{2}$, the proof of $\gamma_{1}=\gamma_{2}$ is similar.

Assume that $\alpha_{1}>\alpha_{2}$. Then from (6.9) one has

$$
\begin{align*}
Q(u, v) & =:\left(\alpha_{1}-\alpha_{2}\right) C^{+}(u, v)+l_{1} G_{1}(u, v) \\
& =\left(\gamma_{2}-\gamma_{1}\right) C^{-}(u, v)+l_{2} G_{2}(u, v) \tag{6.10}
\end{align*}
$$

Thus for almost all $(u, v) \in[0,1]^{2}$,

$$
\frac{\partial}{\partial u} Q(u, v)=\left(\gamma_{2}-\gamma_{1}\right) I_{\{u+v-1>0\}}+l_{2} \frac{\partial}{\partial u} G_{2}(u, v)
$$

Applying (2.4) and by the above equation, one can derive that the comonotonic factor $\alpha_{Q}$ of $Q$ satisfies that

$$
\alpha_{Q}=l_{2} \alpha_{G_{2}}
$$

Since $\alpha_{1}>\alpha_{2}$, from (6.10) we know that the comonotonic factor $\alpha_{Q}>0$. Then the above equation leads to that $\alpha_{G_{2}}>0$, which contradicts to the indecomposable assumption on $G_{2}$. Thus $\alpha_{1} \leq \alpha_{2}$, and similarly $\alpha_{1} \geq \alpha_{2}$. Hence $\alpha_{1}=\alpha_{2}$ holds.

Combining (a) and (b) with the fact that (1.2) holds with $\alpha=\alpha_{C}, \beta=$ $\beta_{C}, \gamma=\gamma_{C}, l_{C}=1-\alpha_{C}-\beta_{C}-\gamma_{C}$ proved before, the first part of the theorem is proved. Now we finish the proof of the theorem.

## 7 Mathematical derivation of Section 3

In this section we will prove some results used in Section 3.
Proposition 7.1. For the optimization problem

$$
s\left(b_{1}^{*}, b_{2}^{*}\right)=\min _{\left\{b_{1}, b_{2}\right\}} s\left(b_{1}, b_{2}\right),
$$

the optimal parameters satisfy (3.3) and (3.4).
Proof. Differentiating $s\left(a_{1}, a_{2}\right)$ with respect to $a_{1}$ and $a_{2}$, respectively, the optimal $b_{1}^{*}$ and $b_{2}^{*}$ should satisfy

$$
\begin{align*}
\int_{0}^{1} \int_{0}^{1} & {\left[G(u, v)-\left(b_{1}^{*} \min \{u, v\}+b_{2}^{*} \max \{u+v-1,0\}+\left(1-b_{1}^{*}-b_{2}^{*}\right) u v\right)\right] } \\
& \cdot(\min \{u, v\}-u v) d u d v=0  \tag{7.1}\\
\int_{0}^{1} \int_{0}^{1} \quad & {\left[G(u, v)-\left(b_{1}^{*} \min \{u, v\}+b_{2}^{*} \max \{u+v-1,0\}+\left(1-b_{1}^{*}-b_{2}^{*}\right) u v\right)\right] } \\
& \cdot(\max \{u+v-1,0\}-u v) d u d v=0 \tag{7.2}
\end{align*}
$$

i.e.,

$$
b_{1}^{*} A_{1}+b_{2}^{*} B_{1}=D_{1}, \quad b_{1}^{*} B_{1}+b_{2}^{*} C_{1}=E_{1}
$$

where

$$
\begin{aligned}
A_{1} & =\int_{0}^{1} \int_{0}^{1}(\min \{u, v\}-u v)^{2} d u d v \\
B_{1} & =\int_{0}^{1} \int_{0}^{1}(\max \{u+v-1,0\}-u v)(\min \{u, v\}-u v) d u d v \\
C_{1} & =\int_{0}^{1} \int_{0}^{1}(\max \{u+v-1,0\}-u v)^{2} d u d v \\
D_{1} & =\int_{0}^{1} \int_{0}^{1}(G(u, v)-u v)(\min \{u, v\}-u v) d u d v \\
E_{1} & =\int_{0}^{1} \int_{0}^{1}(G(u, v)-u v)(\max \{u+v-1,0\}-u v) d u d v
\end{aligned}
$$

By Cramer's rule,

$$
b_{1}^{*}=\frac{D_{1} C_{1}-B_{1} E_{1}}{A_{1} C_{1}-B_{1}^{2}}, \quad b_{2}^{*}=\frac{A_{1} E_{1}-B_{1} D_{1}}{A_{1} C_{1}-B_{1}^{2}}
$$

In fact, after simple calculations, we have

$$
\begin{aligned}
& \int_{0}^{1} \int_{0}^{1}(\min \{u, v\})^{2} d u d v=\frac{1}{6}, \quad \int_{0}^{1} \int_{0}^{1}(u v)^{2} d u d v=\frac{1}{9} \\
& \int_{0}^{1} \int_{0}^{1}(\max \{u+v-1,0\})^{2} d u d v=\frac{1}{12}, \quad \int_{0}^{1} \int_{0}^{1}(\min \{u, v\} u v) d u d v=\frac{2}{15}, \\
& \int_{0}^{1} \int_{0}^{1}(\max \{u+v-1,0\} u v) d u d v=\frac{11}{120} \\
& \int_{0}^{1} \int_{0}^{1}(\min \{u, v\} \max \{u+v-1,0\}) d u d v=\frac{5}{48}
\end{aligned}
$$

Plugging above results into $A_{1}, B_{1}, C_{1}$ yields $A_{1}=\frac{1}{90}, B_{1}=-\frac{7}{720}, C_{1}=\frac{1}{90}$. Hence

$$
A_{1} C_{1}-B_{1}^{2}=\frac{1}{34560}
$$

So (3.3) and (3.4) are obtained.

Next we give the proof of Theorem 3.1.
Proof of Theorem 3.1: The minimum is

$$
\begin{aligned}
& s\left(b_{1}^{*}, b_{2}^{*}\right)=\int_{0}^{1} \int_{0}^{1}[(G(u, v)-u v) \\
& \left.\quad-\left(b_{1}^{*}(\min \{u, v\}-u v)+b_{2}^{*}(\max \{u+v-1,0\}-u v)\right)\right]^{2} d u d v
\end{aligned}
$$

Then due to the conditions (7.1) and (7.2), we have

$$
\begin{aligned}
& s\left(b_{1}^{*}, b_{2}^{*}\right) \\
&=\int_{0}^{1} \int_{0}^{1}(G(u, v))^{2} d u d v+\left(1-b_{1}^{*}-b_{2}^{*}\right) \int_{0}^{1} \int_{0}^{1} u^{2} v^{2} d u d v \\
&+\left(b_{1}^{*}+b_{2}^{*}-2\right) \int_{0}^{1} \int_{0}^{1} G(u, v) u v d u d v \\
&-b_{1}^{*} \int_{0}^{1} \int_{0}^{1} G(u, v) \min \{u, v\} d u d v \\
&-b_{2}^{*} \int_{0}^{1} \int_{0}^{1} G(u, v) \max \{u+v-1,0\} d u d v \\
&+b_{1}^{*} \int_{0}^{1} \int_{0}^{1} u v \min \{u, v\} d u d v+b_{2}^{*} \int_{0}^{1} \int_{0}^{1} u v \max \{u+v-1,0\} d u d v
\end{aligned}
$$

The above value can be written as

$$
\int_{0}^{1} \int_{0}^{1}(G(u, v))^{2} d u d v+\frac{1}{12}-2 \int_{0}^{1} \int_{0}^{1} G(u, v) \max \{u+v-1,0\} d u d v+M
$$

where

$$
M=-384 A_{2}^{2}+\left(1440 C_{2}-672 B_{2}\right) A_{2}+1440 B_{2} C_{2}-1440 C_{2}^{2}-384 B_{2}^{2}
$$

and

$$
\begin{aligned}
A_{2} & =\int_{0}^{1} \int_{0}^{1} G(u, v) C^{+}(u, v) d u d v-\frac{1}{48} \\
C_{2} & =\int_{0}^{1} \int_{0}^{1} G(u, v) C^{\perp}(u, v) d u d v-\frac{1}{120} \\
B_{2} & =\int_{0}^{1} \int_{0}^{1} G(u, v) C^{-}(u, v) d u d v
\end{aligned}
$$

It is easy to check that $M$ is a negative definite quadratic form because its eigenvalues are $0,-1$ and -45 . Due to the fact that $G(u, v)$ is a copula, so $G(u, v) \geq \max \{u+v-1,0\}$, hence the theorem holds.

## 8 Conclusions

This paper presents a convex decomposition in terms of a comonotonic, an independent, a countermonotonic, and an indecomposable part for bivariate copula, and provides deep insights into the dependence structure for bivariate random variables. For the indecomposable part, the approximation by a convex sum of comonotonic copula, independent copula, and countermonotonic copula is discussed, moreover the approximation error bound is provided. This paper also briefly introduces this result's applications in insurance and finance, and gives some application ideas in these fields. Our method allows us to deal with more complicated copula by focusing on its convex approximation.

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