Portfolio optimization problems with linear programming models

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Abstract

In this paper, we discuss four models proposed by Konno, Cai, Teo and Markowitz respectively. Two groups of data (one from 33 securities over 72 months, the other from 63 securities over 120 months) are used to examine these models. Efficient frontiers are presented. The utility levels in the four models do not decrease at the same rate with the change of the risk-aversion factor. Cai's model provides the highest utility value and Markowitz's provides the lowest one in most cases. When the expected returns are confronted with the true ones at the end of a 10-month period, Markowitz's and Konno's models seem to have similar tendencies while Cai's and Teo's models seem to have similar tendencies, and the four models get higher true wealth compared with Nikkei 225 and Nikkei 500 index respectively in most cases.

Keywords: Portfolio selection; Mean absolute deviation; Linear programming; Quadratic programming

1. Introduction

Markowitz (1952) gives a mean-variance method in solving portfolio selection problems. Such a method has been regarded as a milestone in finance. After his work, some different risk control models are proposed and people have paid more attention to the characteristics of those models. The comparison of these models is essential for the investors to choose appropriate one to construct their portfolios. Therefore the way to distinguish these models and state precisely the advantages and disadvantages for them is such a meaningful work that many experts are

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concerned with it. Much work has been done. Here, we mention some of those important findings.

Classical mean variance method employs the expected return to measure the return of a portfolio, and use variance to measure its risk. But there are many researchers and traders who may not be convinced that the covariance is an appropriate risk measure. They assume that the ordinary investors consider its distribution of risk may not be symmetric. In most cases, a little loss will make one very sad, while the considerably high profit can make one very happy. This implies that the classical mean variance model may serve to be some approximation to the complex portfolio problems that all investors encounter. Hence, experts in the financial area exert all possible efforts to present some new risk models and try to meet the needs of different investors. For example: the lower semi-variance model, lower semi standard deviation model (Markowitz, 1959), mean absolute deviation model (Konno and Yamazaki,1990,1991), mean semi absolute deviation model (Konno, 1991; Ogryczak and Ruszczynski, 1999), below target risk model (Fishburn, 1977), maximim model (Young, 1998), minimax model (Cai et al., 2000; Teo and Yang, 2001) and so on.

It is assumed that there are *n* assets in the market. Let R_i be the random return rate of the asset *i*, and x_i be the money allocated to asset *i*, i = 1, ..., n. The return rate of a portfolio R(x) is denoted by $\sum_{i=1}^{n} R_i x_i$, and the expected return rate of asset *i* is denoted by r_i . Denoting by $r(x) = E(R(x)) = (r_1, ..., r_n)$, then the variance is $V(x) = E[R(x) - r(x)]^2$, and the standard deviation is $\sigma(x) = \sqrt{V(x)} = \sqrt{E[R(x) - r(x)]^2}$.

Definition 1 The lower semi-variance is defined as $V_{-}(x) = E[(R(x) - r(x))_{-}]^2$, where $(a)_{-} = \max\{0, -a\}$, and denotes this risk function by LSV.

Definition 2 The lower semi standard deviation is defined as $\sigma_{-}(x) = \sqrt{E[(R(x) - r(x))_{-}]^2}$, and denotes this risk function by *LSSD*.

It is pointed out that when the distribution of the return rate is skewness, that is, $\sigma(x) \neq \frac{1}{2}\sigma(x)$, LSV seems to be more preferable than variance or standard deviation to measure the risk (Markowitz, 1952). Obviously, for those who employ LSV and LSSD, it will be natural to think risk may exist when future return does not exceed the expected return. This idea has got firmly fixed among most investors who are risk aversion. But when R(x) is normally distributed, since these two risk models are similar to the variance model, little importance is placed on LSV and LSSD.

Definition 3 If ρ is the return rate that the investors would expect to have, then the

kth-order target risk is defined as $BT_k(\rho, x) = E[(\rho - R(x))_-^k]^{\frac{1}{k}}$.

This risk measure will be fit into the MEU principle (Fishburn,1977). For any k, $BT_k(\rho, x)$ is convex. When k = 1, 2, the portfolio model can be transformed into problems of linear programming and quadratic programming respectively, which is extremely important in proceeding the real computation.

Konno and Yamazaki (1990,1991) present Mean Absolute Deviation (MAD) model.

Definition 4 Absolute deviation is defined as follows

$$l_1(x) = E \left| \sum_{j=1}^n R_j x_j - E[\sum_{j=1}^n R_j x_j] \right|.$$

The main characteristic of this model is that the risk of a portfolio is measured by the absolute deviation of the return rate of assets instead of the variance. Much attention has been focused on this risk function because the portfolio optimization problem with l_1 risk function can be converted into a scalar parametric linear programming problem. Hence, the implementation of the portfolio optimization with this model can be easily obtained. Simplicity and computational robustness are perceived as one of the most important advantages of the MAD model. Till now, many excellent properties of this model have been found and some of them are referred to here..0

It is pointed out that the MAD model takes on an opportunity to make a more specific model such as the downside risk because absolute deviation may be regarded as a measure of the downside risk (Konno,1990; Feinstein and Thapa, 1993).

It is known that if the return is multivariatly, normally distributed, the minimization of the MAD provides similar results as the classical Markowitz formulation, and minimization of MAD is equivalent to maximization of the expected utility under risk aversion (Rudolf et al, 1999).

Markowtiz model has been criticized as not being consistent with axiomatic models of preferences for choice under risk because it does not depend on a relation of stochastic dominance (Whitemore and Findlay, 1978; Levy, 1992). In contrast, the MAD model is consistent with the second degree stochastic dominance, provided that the trade-off coefficient between risk and return is bounded by a certain constant(Ogryczak, 1997).

Ogryzak and Ruszczynki (1999, 2001) proved that the most optimal solution in efficient frontier of MAD model satisfies the MEU principle no matter how (R_1, \ldots, R_n) is distributed. At the same time, the capital asset pricing model for the l_1 risk model is derived by Konno (1991) where the risk function is assumed to be differentiable at the market portfolio. Without imposing differentiability on the l_1 risk function, equilibrium relations were given by Konno and Shirakawa (1994).

Moreover, since the optimization problem with MAD is always transformed into a linear programming problem, the model can easily be extended to a frictional case (Konno and Wijayanayake, 1999, 2001a, 2001b and 2002), while the mean variance model may be more difficult for these cases. Since there are so many advantages in the MAD model, it is worth discussing and considering its extension.

Cai et al (2000) gives a new risk model based on MAD. He introduces the minimax rule in the portfolio selection model.

Definition 5 The maximum absolute deviation risk model l_{∞} is given as

$$l_{\infty}(x) = \max_{1 \leq j \leq n} E \left| R_j x_j - E(R_j) x_j \right|.$$

In this model, the investor is assumed to minimize the maximum of individual risk. The explicit analytical solution for the model is presented and the entire efficient frontier is also plotted. The author points out that such a risk model is very conservative and it does not explicitly involve the covariance of the asset returns.

Definition 6 The alternative l_{∞} risk function is defined as:

$$H_{\infty}^{T}(x) = \frac{1}{T} \sum_{t=1}^{T} \max_{1 \le i \le n} E |R_{it}x_{i} - r_{it}x_{i}|,$$

where R_{it} is random variables and r_{it} is the expected value of R_{it} , for i = 1, ..., n, t = 1, ..., T. This function is an extension of $l_{\infty}(x)$, and it is assumed that the available historical data are split into T periods. In each period, the individual absolute deviation with respect to the expected value of the period is calculated. The total risk of the portfolio is taken as the average of the maximum (over all assets) of these individual absolute deviations over all periods.

It is worth noting that Papahristodoulou and Dotzauer (2004) compared Markowitz's QP model, Konno's MAD model and Young's Maximum model. They found that the maximin formulation yields the highest return and risk, while the QP formulation provides the lowest risk and return. And it is also pointed out that the minimization of mean abosolute deviation is close to the QP formulation. They conclude that the maximum portfolios seem to be the most robust of the three models when comparing the expected returns with the true ones at the end of a 6-month period.

In this paper, we compare Cai's and Teo's model with Konno's model and Markowitz's model by employing the similar method used by Papahristodoulou and Dotzauer. The reason we consider these four models is that l_{∞} and $H_{\infty}^{T}(x)$ are new risk models based on l_{1} . The difference of these three models is an interesting and meaningful problem for both the researchers and the investors. At the same time, in order to observe the difference of these three models with classical MV model, we also take Markowitz's model into consideration. We find that most of the time, Markowitz's model has the similar tendency to Konno's model and Cai's model has similar tendency to Teo's model. Some interesting results in detail are given later.

The organization of the paper is as follows. In section 2, the four models are presented in detail and the other three models except Markowitz's model, are transformed into linear forms. In section 3, two groups of data are employed to test the four models in four respects which are efficient frontier, utility value, true performance of the four models by using real stock data and computational speed. Computational results and some figures are given in this section. The conclusion and future work are given in section 4.

2. Model description

In this section, we will describe the above four models in detail and each of Konno's, Cai's and Teo's model will be transformed into a linear programming problem respectively.

Let M_0 be the initial wealth the investor holds, and ρ be the return rate the investor required. Denote by μ_i the maximum amount the investor wants to invest in asset i, i = 1, ..., n. It is assumed that the short selling is not permitted, that is, $x_i \ge 0, i = 1, ..., n$. Denote by

$$S = \{x = (x_1, \cdots, x_n) : \sum_{j=1}^n r_j x_j \ge \rho M_0, \sum_{j=1}^n x_j = M_0, 0 \le x_j \le \mu_j, \ j = 1, \dots, n\}$$

Model 1 Konno's model

min
$$w(x) = E \left| \sum_{j=1}^{n} R_j x_j - E \left(\sum_{j=1}^{n} R_j x_j \right) \right|$$

s.t. $x \in S$

Since the objective function is not linear, we follow Konno and Yamazaki's method and express this model in the following way (Konno and Yamazaki,1991):

min
$$w(x) = \frac{1}{T} \sum_{t=1}^{T} y_t$$

s.t. $y_t \ge \sum_{j=1}^{n} (r_{jt} - r_j) x_j, \quad t = 1, \dots, T$
 $y_t \ge -\sum_{j=1}^{n} (r_{jt} - r_j) x_j, \quad t = 1, \dots, T$
 $x \in S$

Here r_j is the expected return of *jth* stock. r_{jt} is the return rate of *jth* stock during period *t*. It is worth noting that we need not to estimate the variance-covariance matrix for this model and the size of the constraints can be easily controlled by the number of the period.

Model 2: Cai's model

min
$$l_{\infty}(x) = \max_{j} E|R_{j}x_{j} - r_{j}x_{j}|$$

s.t. $x \in S$

This model can also be transformed into the following linear form (The proof is in Appendix.):

min
$$y$$

s.t. $q_j x_j \leq y \ j = 1, \dots, n$
 $x \in S$

where $q_j = E|R_j - r_j|$, j = 1, ..., n, which is the expected absolute deviation of R_j from its mean. Obviously, if the distribution of each random variable R_j is given, this function is explicitly determined. Historical data can also be used to estimate r_j and q_j . The l_{∞} model and the related techniques are easy to manipulate and implement in practice. Moreover, the selection of the optimal portfolio does not involve the correlations among stocks, which is similar to Konno's model, and the number of constraints for this model is determined by the number of stocks.

Model 3: Teo's model

min
$$H_{\infty}^{T}(x) = \frac{1}{T} \sum_{t=1}^{T} \max_{j} E|R_{jt}x_j - E(R_{jt})x_j|$$

s.t. $x \in S$

For this model, a capital asset pricing model between the market portfolio and each individual return is established by using a nonsmooth optimization method. This model can be transformed into the following linear form. The proof is in the Appendix.

min
$$\frac{1}{T} \sum_{t=1}^{T} y_t$$

s.t. $a_{jt} x_j \leq y_t, \quad t = 1, \dots, T, j = 1, \dots, n$
 $x \in S$

where $a_{jt} = E|R_{jt} - E(R_{jt})|$, j = 1, ..., n, t = 1, ..., T. Obviously, the size of the constraints is determined by the number of the stocks and the number of the periods. Obviously, if n or T becomes large, the computational speed of solving this model may certainly get slow. At last, we recall the classical model: Markowitz' model

min
$$V(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j$$

s.t. $x \in S$

The characteristic of this model is that it is a quadratic programming. Noting that the objective function is related to variance-covariance matrix, so when investors choose such a model, it means they should first take a certain amount of time to calculate this matrix. For instance, if there are 200 securities, i.e., n = 200, we have to calculate a variance-covariance matrix of 20100 combinations. Such a time-consuming calculation is unnecessary for those above 3 models. But with the rapid development of computational technologies, such a work is no longer as difficult as before. We also notice that the minimization of the variance-covariance matrix might lead to inefficient portfolios unless one sets explicitly an expected return. Related description is given by Papahristodoulou and Dotzauer(2004).

3. Data and computational results

We will use two groups of data to examine all models from several points of view. First, we will give the efficient frontier of the four models. Efficient frontier analysis of the portfolio selection is important as it gives clearly a geometric scope of the relation between the return and the risk. The second, the utility values of the optimal portfolio are considered. A useful criterion to determine which portfolio should be selected for different risk measures is to recognized by observing various utility functions (Sharpe et al, 1999). The third, we will compare the true performance with the expected value of the four models. Constructing the portfolio given by the models, we examine the true wealth of these portfolios during 10 months. Finally, we will consider the computational speed. Computational speed is also an important factor for investor to consider when they decide to choose a suitable model. A model will not be selected if the computational speed is too slow, and a good investment chance is often lost as time goes by.

The first group consists of 33 stocks traded in Nikkei 225, using monthly returns from January 1995 to December 2000. We note one reason not to include data after 2000 is, as seen in most places in the world, that the return dropped down very sharply over the last 4 years. The second group consists of 63 stocks traded in Nikkei 500, whose monthly returns are employed from January 1991 to December 2000. The second group consists of more stocks and the period is longer than that in the first group. The criterion to select the stocks in our examination is described as follows:

1. Since the portfolios are examined on the basis of the historical data, those with negative average returns over the examined period are excluded.

2. Those companies which were not on the list at the starting point and entered the Nikkei 225 (group 1) or Nikkei 500 (group 2) at different dates afterwards are excluded.

3. Those companies which were on the list at the starting point and not in the list at the end of the examined period are excluded.

4. Those stocks which have the positive returns but with too small values are excluded.

5. Large companies are taken preference over small and medium-sized companies.

We assume that an investor has the initial wealth whose value is equal to 1 unit and require various monthly returns. For group 1 and 2, the required return rate ρ is set from 1 to 2.2%. The investors also wish that each asset would not receive more than 60% of their budget. Table 1 and Table 2 give the expected returns and risks for various values of ρ for group 1 and group 2 respectively.

Teo	Cai	Konno	Markowitz		ρ
0.01277	0.012636	0.014332	0.013569	ER	1.00%
0.006083	0.002224	0.028913	0.036971	σ	
0.013	0.0130	0.014332	0.013569	ER	1.30%
0.006138	0.002399	0.028913	0.036971	σ	
0.015	0.01500	0.01500	0.01500	\mathbf{ER}	1.50%
0.009095	0.004031	0.029106	0.037565	σ	
0.01800	0.01800	0.01800	0.01800	ER	1.80%
0.019092	0.011259	0.033365	0.042217	σ	
0.02	0.020	0.02	0.02	ER	2.0%
0.028656	0.020149	0.039312	0.048263	σ	
0.022	0.02200	0.02200	0.02200	ER	2.2%
0.043038	0.033994	0.054864	0.068506	σ	
105	34	105	33	Variables	
2378	35	146	2	Constraints	

Table 1 Monthly average expected returns for group 1

ρ		Markowitz	Konno	Cai	Teo
1-1.20%	\mathbf{ER}	0.012	0.012	0.014526	0.013076
	σ	0.0391	0.0279	0.001148	0.007137
1.60%	ER	0.016	0.016	0.016	0.016
	σ	0.043143	0.031377	0.001377	0.008112
1.80%	\mathbf{ER}	0.018	0.018	0.018	0.018
	σ	0.045992	0.033848	0.001797	0.009732
2.00%	\mathbf{ER}	0.020	0.020	0.020	0.020
	σ	0.049124	0.03682	0.002374	0.011724
2.20%	\mathbf{ER}	0.022	0.022	0.022	0.022
	σ	0.052745	0.040187	0.003122	0.013959
	Variables	63	183	64	183
	Constraints	2	242	65	7562

Table 2 Monthly average expected returns for group 2 $\,$



Figure 1: Efficient frontier

(I) Efficient frontier of the models

Figure 1 shows that the curves of these four models' efficient frontier are similar for both groups. We can find the following interesting results when comparing two Tables.

1. Markowitz' curve is closer to Konno's. This result is also given by Papahristodoulou and Dotzauer. And Cai's curve is closer to Teo's. For the same expected return, four models show

different risk values. Thus, the risk values for $\sigma(x)$ and $l_1(x)$ are always higher than those for $l_{\infty}(x)$ and $H_{\infty}^T(x)$.

2. It is found that for the second group which consists of 63 stocks, Cai's model show lower risk values than that in group 1. For example, for Cai's model in group 1, the risk varies from 0.002224 to 0.033994 when the required return rate ρ varies from 1 to 2.2%, while in group 2, risk varies from 0.001148 to 0.003122. But such a trend can not be observed for the other 3 models. Hence it may be considered that Cai's model is more sensitive to the diversification of the risk when the number of stocks increases.

3. In Table 1, compared with Cai's model and Teo's model, Markowitz's and Konno's models are more robust because the portfolio derived from them remains unchanged for ρ ranging from 1 to 1.3569% and 1.4332% respectively. But in Table 2, the same result can also be given for Cai's and Teo's models. The optimal portfolio derived from them remains unchanged from 1 to 1.4526 and 1.3076% respectively. Hence, we would claim that which model will be more robust should not be concluded by only observing one group of data. Papahristodoulou and Dotzauer thought maximum model is more robust by using one group of data. Such a result may not be so confident as it looks.

4. It should be noted that Markowitz's model provides higher risk than Konno's model, and Teo's model provides higher risk than Cai's model at any required return rate for both groups.

(II) Utility of four models.

The following simple form (mentioned by Sharp et al.) as a standard method to represent the investor's indifference curves in a mean-variance context:

$$U = E(R) - w\sigma^2$$

where U is the level of utility, E(R) is the expected return and w is a positive constant which indicates the investor' risk aversion (Sharp et al,1999). Obviously, if w = 0, the utility level that the specific portfolio provides is independent of its risk. If the value of w approaches infinity, this means the investor will allocate all the money to risk-less asset.

Figure 2 summarizes the utility levels for various values of risk factor w and different required return rate ρ for group 1 and group 2, respectively.

It may be found that except for w = 0, Markowitz's model provides the lowest utility, and Cai's model provides the highest one for both groups. Moreover, we note that utility levels in the four models do not drop at the same rates with one another. Utility in Markowitz's model decreases more sharply, compared with the other three models. It is worth noting that utility in Cai's and Teo's model seems to decreases very slowly as w varies from 0 to 1. Hence, it may be difficult for the investor to choose a portfolio according to the utility value of Cai's and Teo's model because the utility value does not show an obvious preference as w varies.

(III)Wealth over ten-months period.

We will examine the true performance of these models for the next 10 months and compare the results with the expected values, Nikkei 225 index (for group 1) and Nikkei 500 index (for group 2) respectively. We assume that the investors are confident of these models and wish to construct their portfolio at the end of December 2000. Obviously, every investor wishes to get positive returns but because of the decline of the stock market from 2000, the true performance is poor. All models give positive expected returns as we expected, but the true monthly returns were negative. Table 3-4 show this clearly.

ho		Markowitz	Konno	Cai	Teo
0.01	True	0.87338	0.90733	0.78625	0.80205
	Expected	1.1841	1.1909	1.1924	1.19
0.018	True	0.88371	0.9058	0.79366	0.80206
	Expected	1.1953	1.1953	1.1953	1.1953
0.02	True	0.86485	0.86547	0.7895	0.7955
	Expected	1.219	1.219	1.219	1.219
0.022	True	0.82294	0.81706	0.7441	0.78427
	Expected	1.2431	1.2431	1.2431	1.2431
0.024	True	0.78402	0.79043	0.651	0.70518
	Expected	1.2677	1.2677	1.2677	1.2677
0.026	True	0.77031	0.77785	0.60457	0.66726
	Expected	1.2926	1.2926	1.2926	1.2926
0.028	True	0.7552	0.7793	0.6208	0.6601
	Expected	1.3180	1.3180	1.3180	1.3180

Table 3: Wealth of four models for 33 assets

ho		Markowitz	Konno	Cai	Teo
0.012	True	0.91895	0.89162	0.89065	0.97721
	Expected	1.1046	1.1053	1.1551	1.1387
0.016	True	0.83258	0.84852	0.84026	0.8613
	Expected	1.172	1.172	1.172	1.172
0.018	True	0.82267	0.81737	0.81035	0.79597
	Expected	1.1953	1.1953	1.1953	1.1953
0.02	True	0.81296	0.8154	0.78089	0.69809
	Expected	1.219	1.219	1.219	1.219
0.022	True	0.79653	0.82688	0.75527	0.67099
	Expected	1.2431	1.2431	1.2431	1.2431

Table 4: Wealth of four models for 63 assets

Figure 3 reflects the change of the wealth during the 10 months for group 1 and group 2.

For group 1, we can summarize that:

1. The curves of wealth for Markowitz's and Konno's model show similar change, while the curves of wealth for Cai's and Teo's model are similar.

2. The wealth values for Markowitz's and Konno's are always higher than that of Cai's and Teo's each month.

3. At the end of 10 month, Markowtiz's and Konno's model always get higher wealth than that of Nikkei 225 index. While for Cai's and Teo's model, it is found that when the required rate is not so high, for example, $\rho = 1\%$ or 2%, these two models get higher wealth than that of Nikkei 225, but when $\rho = 2.8\%$, this characteristic disappears.

4. During the 10 months, it is found that when the required return is low, (for example, when $\rho = 1\%, 2\%$), the wealth in the 10 months is higher than that of the Nikkei 225 index in most times for Markowitz's and Konno's models. But when ρ becomes higher, for example, ($\rho = 2.8\%$), such a tendency disappears.

For group 2, it is obvious that:

1. All four models always get higher wealth than that of Nikkei 500 index at the end of 10 month.

2. During the 10 months, it is found that for the four models, when the required return is low, (for example, when $\rho = 1.2\%, 1.6\%$), the wealth in the 10 months is higher than that of

the Nikkei 500 index in most cases. But when ρ becomes higher, (for example, $\rho = 2\%$), such a tendency disappears for Teo's model.

It is worth noting that if Markowitz's and Konno's results, as well as Cai's and Teo's results are plotted in the respective planes, the similarity for the tendency of the change of the wealth is identified with its own case. This is also observed in group 1. Figure 4 shows this result clearly.

(IIII) Computation speed.

We use Matlab 7.0 to calculate all results. It is found that the solution procedure for Teo's model is influenced by the number of stocks and the number of periods we choose. If we consider that every month is counted as a period, the constraints of Teo's model are very large, which make the computation speed slow, especially for 63 stocks. The other three models show no evidence that the number of the stocks and periods influence the speed of the solution procedure.

4. Conclusion

In this paper, we compare 4 models, which are Markowitz's model, Konno's MAD model, Cai's model and Teo's model. Two groups of data from the Tokyo stock markets are employed to compare them in four respects. Efficient frontiers are given and the utility values are considered. Moreover, we construct portfolios according to the models and compare the expected value with the true ones. The computational speed is also discussed. It is found that all the four models have the similar shape of efficient frontier. The utility levels in the four models do not fall at the same rate with the change of the risk-aversion factor. In most cases, Cai's model has the highest utility values and Markowitz' model has the lowest one, and it may be difficult for the investor to choose portfolios according to the utility value of Cai's and Teo's models because the utility value does not show obvious preference as the risk factor varies. Moreover, when the expected returns are faced with the true ones at the end of a 10-month period, Markowitz's model and Konno's model seem to have similar tendencies while Cai's and Teo' models seem to have similar tendencies. At the same time, it is found that at the end of the 10-month period, (both in two groups), four models can get higher true final wealth compared with Nikkei 225 index and Nikkei 500 index respectively in most cases. As for Teo's model, the solution procedure is influenced by the number of stocks and periods.

For future study, we are concerned with the dynamic portfolio employing absolute deviation. It is well known that a classical Mean-variance model has been extended to multiperiod cases (see Li and Ng, 2000; Li et al, 2002), but how about the MAD model? We believe that some extension of the MAD model to a multiperiod case is a very interesting and challenging problem to be solved.

5. Appendix

I. Proof of the two theorems.

Denoted by P_1 and P_2

$$P_1 \begin{cases} \min & \max_j q_j x_j \\ \text{s.t.} & x \in S \end{cases} \qquad P_2 \begin{cases} \min & y \\ \text{s.t.} & q_j x_j \leq y \ j = 1, \dots, n \\ & x \in S \end{cases}$$

Theorem A.1 If x^* is an optimal solution to P_1 , then (x^*, y^*) is an optimal solution to P_2 , where $y^* = \max_j q_j x_j$. On the other hand, if (x^*, y^*) is an optimal solution to P_2 , then x^* is an optimal solution to P_1 .

Proof. If x^* is an optimal solution to P_1 , then (x^*, y^*) is a feasible solution to P_2 , where $y^* = \max_j q_j x_j$. If (x^*, y^*) is not an optimal solution to P_2 , then there exists a feasible solution (x, y) to P_2 , such that $y < y^*$. Noticing that $q_j x_j \leq y$, then $\max_j q_j x_j \leq y < y^* = \max_j q_j x_j^*$. This contradicts to that x^* is an optimal solution to P_1 .

On the other hand, if (x^*, y^*) is an optimal solution to P_2 , then x^* is a feasible solution to P_1 . If x^* is not an optimal solution to P_1 , then there exits a feasible solution x to P_1 , such that $\max_j q_j x_j < \max_j q_j x_j^*$. Denote by $y = \max_j q_j x_j$. Then we have $y = \max_j q_j x_j < \max_j q_j x_j^* \le y^*$. This contradicts that (x^*, y^*) is an optimal solution to P_2 .

The proof is complete.

Denoted by P_3 and P_4 respectively

$$P_3 \begin{cases} \min & \frac{1}{T} \sum_{t=1}^T \max_j a_{jt} x_j \\ \text{s.t.} & x \in S \end{cases} \qquad P_4 \begin{cases} \min & \frac{1}{T} \sum_{t=1}^T y_t \\ \text{s.t.} & a_{jt} x_j \leq y_t \quad t = 1, \dots, T, j = 1, \dots, n \\ & x \in S \end{cases}$$

T

Theorem A.2 If x^* is an optimal solution to P_3 , then (x^*, y^*) is an optimal solution to P_4 , where $y^* = (y_1^*, \ldots, y_t^*, \ldots, y_T^*)$, $y_t^* = \max_{1 \le j \le n} a_{jt} x_j^*$. On the other hand, if (x^*, y^*) is an optimal solution to P_4 , where $y^* = (y_1^*, \ldots, y_T^*)$, then x^* is an optimal solution to P_3 .

Proof. If x^* is an optimal solution to P_3 , then (x^*, y^*) is a feasible solution to P_4 , where $y^* = (y_1^*, \ldots, y_T^*), y_t^* = \max_{1 \le j \le n} a_{jt} x_j^*$. If (x^*, y^*) is not an optimal solution to P_4 , then there exists a feasible solution (x, y), where $y = (y_1, \ldots, y_T)$, to P_4 , such that $\frac{1}{T} \sum_{t=1}^T y_t < \frac{1}{T} \sum_{t=1}^T y_t^*$. Noticing

that $a_{jt}x_j \leq y_t$, then we have

$$\frac{1}{T}\sum_{t=1}^{T}\max_{1\leq j\leq n}a_{jt}x_{j}\leq \frac{1}{T}\sum_{t=1}^{T}y_{t}<\frac{1}{T}\sum_{t=1}^{T}y_{t}^{*}=\frac{1}{T}\sum_{t=1}^{T}\max_{1\leq j\leq n}a_{jt}x_{j}^{*}$$

which contradicts that x^* is an optimal solution to P_3 .

On the other hand, if (x^*, y^*) is an optimal solution to P_4 , where $y^* = (y_1^*, \ldots, y_T^*)$, then x^* is an optimal solution to P_3 . Otherwise, there exists a feasible solution x to P_3 , such that $\frac{1}{T} \sum_{t=1}^{T} \max_{1 \le j \le n} a_{jt} x_j < \frac{1}{T} \sum_{t=1}^{T} \max_{1 \le j \le n} a_{jt} x_j^*$. Denote by $y_t = \max_{1 \le j \le n} a_{jt} x_j$, and $y = (y_1, \ldots, y_T)$. Then we have $1 \sum_{t=1}^{T} \sum_{t=1}^{T} \sum_{1 \le j \le n} 1 \sum_{t=1}^{T} \sum_$

$$\frac{1}{T}\sum_{t=1}^{T} y_t = \frac{1}{T}\sum_{t=1}^{T} \max_j a_{jt} x_j < \frac{1}{T}\sum_{t=1}^{T} \max_j a_{jt} x_j^* \le \frac{1}{T}\sum_{t=1}^{T} y_t^*$$

which contradicts that (x^*, y^*) is an optimal solution to P_4 .

Hence, we complete the proof.

II. The list of the companies.

Group 1: The list of 33 assets in Nikkei 225:

T2531, T4063, T4452, T4505, T4506, T4507, T4519, T4523, T4543, T4901, T5202, T5706, T5801, T6367, T6701, T6702, T6752, T6758, T6762, T6764, T6971, T6976, T7203, T7267, T7270, T7733, T7751, T7752, T8035, T4502, T4503, T6857, T6954.

Group 2: The list of 63 assets in Nikkei 500:

N0000247, N0000488, N0000489, N0000519, N0000525, N0000710, N0000529, N0000531, N0000565, N0000559, N0000592, N0000593, N0000597, N0000617, N0000622, N0013612, N0000697, N0000722, N0000759, N0001382, N0001387, N0001392, N0001393, N0001394, N0001408, N0001451, N0001452, N0000728, 0001458, N0001459, N0001516, N0001738, N0001708, N0002142, N0012655, N0005640, N0024129, N0001680, N0001683, N0070046, N0001711, N0001637, N0028448, N0070201, N0028448, N0070204, N0031596, N0001874, N0002001, N0002003, N0015006, N0005754, N0004459, N0002104, N0017193, N0005290, N0027118, N0014401, N0002031, N0008680, N0007573, N0001736, N0068435.

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Figure 2: Utility



Figure 3: Wealth for group 1 and group 2 $\,$



Figure 4: Wealth