Intensity-based framework for optimal stopping problems

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Abstract

Financial derivatives commonly contain pre-mature termination clauses, which are embedded rights held by the holder or writer. Well known examples of these stopping rights include the early exercise right in American options, callable right in callable securities and prepayment right in mortgage loans. In this paper, we show how to model the mortgagor's prepayment in mortgage loans and issuer's call in American warrant as event risks using the intensity based approach, where the propensity of prepayment or calling is modeled by the intensity of a Poisson process. We illustrate that the corresponding pricing formulation resembles the penalty approximation approach commonly used in the solution of the linear complementarity formulation of an optimal stopping problem. We obtain several theoretical results on the prepayment strategies of mortgage loans and calling polices of American warrants. We also propose robust second order accurate numerical schemes for solving the penalty formulation of an optimal stopping problem.

Keywords: linear complementarity formulation, mortgage prepayment, callable feature, intensity approach, penalty method, event risks

1 Introduction

Mortgage loans and bond contracts are both debt instruments, except that the repayment of the principal in a mortgage loan is amortized over the life of the mortgage while the bond par is usually paid in full at the maturity of the bond. The mortgagor plays a similar role as the bond issuer since both owe the scheduled stream of cash flows on their liabilities. Most mortgage loans contain the embedded prepayment privilege that gives the mortgagor the right to terminate the contract prematurely by paying the remaining principal plus any applicable transaction costs. The studies on the behaviors of prepayment and mortgage termination have been well explored in the literature in the past decades. Prepayment models that include the consideration of burnout effects and macro-economic factors have been proposed for the valuation of different types of mortgage backed securities (Deng *et al.*, 2000; McConnell and Singh, 1994; Schwartz and Torous, 1992; Stanton, 1995). In callable bonds and other callable derivative securities, the embedded callable right entitles the issuer to recall the derivative by paying a pre-set cash amount (call price). There may be some imposed constraints on the calling provision, like the soft and hard call requirements and notice period requirement (Lau and Kwok, 2004). The discussion of the optimal calling policies of callable American warrants and convertible bonds can be found in the papers by Kwok and Wu (2000) and Dai and Kwok (2005). In essence, both the prepayment and callable rights limit the market value of future cash flow liabilities via early termination of the contract through an exchange of future liabilities by an upfront single payment.

The mortgagor's prepayment in mortgage loans and issuer's callable right in callable securities are vivid examples of pre-mature termination clauses that are commonly found in financial derivatives. Assuming that the execution of these rights is optimally chosen by the writer or buyer of these securities, the modeling of early termination clauses can be formulated as optimal stopping problems. By solving the linear complementarity formulation of the pricing model, the optimal stopping rule and derivative price are obtained simultaneously. However, numerous empirical studies have shown that in general these rights would not be exercised optimally following the optimal stopping rules. Market frictions, corporate finance considerations and other factors may affect their "rational" behaviors of exercising the embedded right of pre-mature termination.

The intensity based approach is first developed in credit risk modeling to model the arrival of a default event. Similar intensity based framework has been adopted by Carr and Linetsky (2001) and Szimayer (2004) in the valuation of executive stock options subject to potential early departure of the executive, and by Goncharov (2004) in the valuation of mortgage contracts. Also, Szimayer (2005) studies the valuation of American options in the presence of event risk by modeling the arrival of event risk as the first jump time of a Cox process. Here, we consider prepayment or calling as an event risk and model the propensity of event arrival by the intensity of a Poisson process.

The penalty method is a well known approximation approach for solving the linear complementarity formulation of an optimal stopping problem (Friedman, 1982). In this paper, we show that the intensity of calling or prepayment can be visualized as the penalty parameter in the penalty approximation. When we take the limit of the penalty parameter to infinity, the penalty approximation becomes exact. Without any surprise, infinite value of intensity represents the scenario where the right is exercised following the optimal stopping rule. On the aspect of numerical computation, Forsyth and Vetzal (2002) propose an implicit finite difference scheme with quadratic rate of convergence for valuing American options using the penalty approximation. However, the convergence behaviors of their scheme appear to be quite erratic. Similar penalty approximation approach has been used to price American options whose underlying asset process is modeled by the stochastic volatility model (Zvan *et al.*, 1998) or jump diffusion model (d'Halluin *et al.*, 2004). In addition, Khaliq *et al.* (2005) develop adaptive θ-methods for solving the penalty formulation of pricing models of one-asset and two-asset American options. To improve computational efficiency and convergence behaviors, we construct two modified versions of Forsyth-Vetzal's scheme for solving the penalty formulation of an optimal stopping problem. Our numerical tests reveal that the proposed numerical schemes demonstrate better computational efficiency and convergence behaviors.

This paper is organized as follows. In Section 2, we review the rational prepayment model proposed by Stanton (1995). The Stanton model allows for prepayment with both exogenous and endogenous reasons. By modeling the arrival of the prepayment event by a Poisson process, we derive the governing differential equation of the continuous version of Stanton's model. We then show how the differential equation formulation can be interpreted as the penalty approximation of the linear complementarity formulation of an optimal stopping problem. We analyze the monotonicity property of the mortgage value with respect to the intensity of prepayment. We also examine the impact of the transaction costs on the boundary that separates the prepayment region and non-prepayment region. In Section 3, we propose two versions of second order accurate finite difference schemes for solving the penalty formulation of an optimal stopping problem. Numerical tests were performed to demonstrate the quadratic rate of convergence of the numerical schemes. In Section 4, we model the propensity of issuer's calling of an American warrant by the intensity of a Poisson process. The impact of the intensity of calling on the optimal holder's exercise policy of the American warrant is analyzed. We end the paper with summary and conclusive remarks in the last section.

2 Intensity-based approach of pricing mortgage loans with prepayment option

In this section, we concentrate on the pricing of a single mortgage loan by modeling the prepayment decision process of the mortgagor. In Stanton's prepayment model (1995), the arrival of prepayment event is modeled using the intensity based approach. His model allows two commonly observed "irrational" behaviors: mortgagors may prepay when it is not financially optimal to do so and delay refinancing even prepayment is financially more beneficial.

Under the full "rationality" assumption, each mortgagor minimizes the market value of the mortgage liabilities and acts rationally to exercise the prepayment right. Let L_t denote the present value of the mortgagor's liabilities and $P(t)$ be the remaining outstanding principal of the mortgage loan. The transaction cost is assumed to be proportional to the outstanding principal, which can be written as $P(t)X$, where X is the proportional factor of transaction cost. The total payout $\psi(t)$ by the mortgagor upon prepayment is then equal to $P(t)(1+X)$. The full rationality assumption would lead to the following simple prepayment decision rule: prepay whenever $L_t > \psi(t)$ and not to do so otherwise. Here, the transaction costs should be interpreted in a broader sense. Besides the actual monetary costs, they also include the inconvenience costs, like the burden of going through the whole prepayment procedure.

In our mathematical setup, we assume the absence of arbitrage opportunities in the market so that a risk neutral measure Q exists. The uncertainty of the economy is modeled by a filtered probability space $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t\geq 0}, Q)$, where the σ-algebra $\mathcal{G}_t$ represents all observations available to the mortgagor at time t, Ω is the sample space of all outcomes and Q is a risk neutral measure on $\mathcal{G}$, where $\mathcal{G} \supseteq \cup_{t\geq 0}\mathcal{G}_t$. In our continuous-time diffusion state process setting, we take the interest rate as the single stochastic state variable in the pricing model. Let τ denote the prepayment time of the mortgage loan, which is a positive stopping time on this filtered probability space. Let γ_t denote the intensity of the random prepayment time τ, then γ_t is a $\mathcal{G}_t$-adapted intensity process. We consider another filtration $\mathcal{F}_t$, which is the natural filtration generated by the interest rate process. Since prepayment decision is not driven by the interest rate movement alone, τ is not a $\mathcal{F}_t$-stopping time.

Following the modeling of prepayment behaviors as postulated by Stanton (1995), a mortgagor may prepay his mortgage loan either for exogenous or endogenous reasons. We refer to prepayment due to migration, divorce, etc., those not driven by interest rate considerations, as exogenous. The arrival of exogenous prepayment is modeled as a Poisson process with constant intensity λ. On the other hand, a mortgagor may consider to refinance when $L_t > \psi(t)$. Such prepayment decision is said to be endogenous. The endogenous prepayment is also modeled by a Poisson process with intensity $\rho\mathbf{1}_{\{L_t>\psi(t)\}}$, where ρ is a constant, reflecting the fact that the intensity of endogenous prepayment is zero when $L_t \leq \psi(t)$. The prepayment time τ is the minimum of these two independent random times, so the intensity γ_t of τ is simply the sum of their intensities. The intensity γ_t has dependence on L_t, which can be expressed as

$$\gamma_t = \left\{ \begin{array}{ll} \lambda & \text{if } L_t \leq \psi(t) \\ \lambda + \rho & \text{if } L_t > \psi(t) \end{array} \right. . \tag{2.1}$$

Partial differential equation formulation

Under the risk neutral measure Q, the dynamics of the stochastic short rate r_t is assumed to be governed by the Ito process

$$dr = \mu_r(r,t)\,dt + \sigma_r(r,t)\,dZ, \tag{2.2}$$

where μ_r is the drift, σ_r is the volatility and dZ is the differential of the standard Wiener process. Let $c(t)$ denote the continuous stream of amortized cash flows paid by the mortgagor throughout the contract until termination. We write $L(r,t)$ as the value of the mortgage liabilities and $\gamma(L)$ as the intensity of the prepayment time with dependence on L. The governing equation for $L(r,t)$ can be derived from the following relation

$$rL\,dt = E_t[dL + c(t)\,dt],$$

where E_t is the expectation under Q conditional on the filtration $\mathcal{G}_t$. By Ito's lemma, we have

$$\begin{aligned} rL\,dt &= \frac{\partial L}{\partial t}\,dt + \mu_r \frac{\partial L}{\partial r}\,dt + \frac{\sigma_r^2}{2}\frac{\partial^2 L}{\partial r^2}\,dt \\ &\quad + \gamma(L)[\psi(t) - L]\,dt + c(t)\,dt. \end{aligned}$$

The differential equation for $L(r, t)$ is then obtained as follows

$$\frac{\partial L}{\partial t} + \mu_r \frac{\partial L}{\partial r} + \frac{\sigma_r^2}{2}\frac{\partial^2 L}{\partial r^2} - [r + \gamma(L)]L + c(t) + \gamma(L)\psi(t) = 0. \tag{2.3}$$

Based on the intensity γ_t defined in eq. (2.1), the above governing equation can be succinctly expressed as

$$\begin{aligned} &\frac{\partial L}{\partial t} + \mu_r \frac{\partial L}{\partial r} + \frac{\sigma_r^2}{2}\frac{\partial^2 L}{\partial r^2} - (r + \lambda)L + c(t) + \lambda\psi(t) \\ = \;&\rho \max(L - \psi(t), 0). \end{aligned} \tag{2.4}$$

For an finite value of ρ, the term $\rho \max(L - \psi(t), 0)$ models a sub-optimal policy of endogenous prepayment. The governing equation is seen to be a non-linear differential equation. In the limit $\rho \to \infty$, the prepayment would be immediate when L_t reaches $\psi(t)$ from below.

There is a distinction between the liability to the mortgagor $L(r, t)$ and the fair value of the mortgage $M(r, t)$. The difference arises since the mortgagor pays $P(t)(1 + X)$ at the time of prepayment but the value of the mortgage loan equals the outstanding principal $P(t)$ upon prepayment. The mortgage value is not simply given by the discounted expected value of the cash flow $c(t)$ since prepayment may occur. By following similar argument as above and replacing the payment term $\psi(t)$ in eq. (2.3) by $P(t)$, the governing differential equation for $M(r, t)$ is deduced to be

$$\begin{aligned} &\frac{\partial M}{\partial t} + \mu_r \frac{\partial M}{\partial r} + \frac{\sigma_r^2}{2}\frac{\partial^2 M}{\partial r^2} - (r + \lambda)M + c(t) + \lambda P(t) \\ = \;&\rho[M - P(t)]\mathbf{1}_{\{L > \psi(t)\}}. \end{aligned} \tag{2.5}$$

Given the known solution to $L(r, t)$, the above differential equation for $M(r, t)$ is linear.

The outstanding principal $P(t)$ can be obtained by solving the following differential equation

$$c(t)\,dt = -dP(t) + m(t)P(t)\,dt,$$

where $m(t)$ is the mortgage rate. Using the initial condition: $P(0) = P_0$. The solution for $P(t)$ is easily found to be

$$P(t) = P_0 e^{\int_0^t m(u)\,du} - \int_0^t C(s) e^{\int_s^t m(u)\,du}\,ds. \tag{2.6}$$

The relation between $C(t)$ and $m(t)$ can be established by observing the terminal condition: $P(T) = 0$. If the amortized cashflow is taken to be independent of time, then the constant cashflow c is related to the fixed mortgage rate m_0 (set at initiation) by

$$c = \frac{m_0}{1 - e^{-m_0 T}} P_0. \tag{2.7}$$

Penalty approximation

Assuming the mortgagor to be fully rational, the optimal endogenous strategy is to exercise the prepayment right immediately when the liability L_t rises to the level $\psi(t)$. This would mean an infinite value for the intensity ρ. The mortgage pricing model then becomes an optimal stopping problem. Let $\mathcal{L}$ denote the operator defined as

$$\mathcal{L} = \mu_r \frac{\partial}{\partial r} + \frac{\sigma_r^2}{2} \frac{\partial^2}{\partial r^2} - (r + \lambda).$$

The linear complementarity (variational inequalities) formulation of the optimal stopping problem is given by

$$\begin{gathered}
\frac{\partial L}{\partial t} + \mathcal{L}L + c(t) + \lambda\psi(t) \geq 0 \\
L - \psi(t) \leq 0 \\
\left[\frac{\partial L}{\partial t} + \mathcal{L}L + c(t) + \lambda\psi(t)\right][L - \psi(t)] = 0.
\end{gathered} \tag{2.8}$$

From the theory of variational inequalities of free boundary problems (Friedman, 1982), the differential equation formulation in eq. (2.4) can be visualized as the penalty approximation to the linear complementarity formulation in eq. (2.8). Here, the parameter ρ in the penalty term has a specific financial interpretation in the prepayment model. It is interpreted as the intensity of endogenous prepayment.

Separating boundary

Recall that a necessary condition for endogenous prepayment is given by $L_t > \psi(t)$. As it is obvious that $\dfrac{\partial L}{\partial r} \leq 0$, so the critical threshold on the interest rate r that separates $L_t > \psi_t$ and $L_t < \psi_t$ in the (r,t)-plane can be defined as

$$r^*(t) = \min\{r(t); \quad L_t < \psi(t)\}. \tag{2.9}$$

How would the separating boundary $r^*(t)$ depend on the intensity of prepayment and transaction cost? First, we would like to establish the monotonicity property of the value function with respect to the penalty parameter ρ. We then present the lemma that states the monotonicity property of $r^*(t)$ with respect to the proportional factor of transaction cost X.

Lemma 1

Let the liability value function $L_i, i = 1, 2$, be the solution to the following penalty formulation

$$\left(\frac{\partial}{\partial t} + \mathcal{L}\right) L_i + f_i(t) = \rho_i \max(L_i - \psi, 0), \quad i = 1, 2,$$

sharing the same set of initial-boundary conditions. Here, ρ_1 and ρ_2 are constant penalty parameters, and $f_1(t)$ and $f_2(t)$ are source terms. Suppose $\rho_1 \geq \rho_2 > 0$ and $f_1(t) \leq f_2(t)$ for all t, then the value functions L_1 and L_2 observe

$$L_1 \leq L_2 \quad \text{for all values of } \rho \text{ and } t.$$

The proof of Lemma 1 using the comparison principle in partial differential equation theory is presented in Appendix A. Using the monotonicity result established in Lemma 1, it becomes straightforward to show Lemma 2.

Lemma 2

If $X_1 < X_2$, then $r^*(t; X_1) \geq r^*(t; X_2)$.

The financial interpretation of the above lemma is quite obvious. With a higher transaction cost, the interest rate has to be lowered further in order to increase the value of liability L_t to the level of prepayment payout $\psi(t)$. The proof of Lemma 2 is presented in Appendix B.

In Figures 1 and 2, we show the plot of the separating boundary $r^*(t)$ [corresponding to $L_t = \psi(t)$] against time to expiry $T - t$ with varying values of the penalty parameter ρ and proportional transaction cost factor X, respectively. In the calculations, we assume that the interest rate dynamics under Q is given by

$$dr = [0.29368(0.07935 - r) + 0.12165r]\, dt + 0.11425\sqrt{r}\, dZ.$$

Also, the amortized cashflow is assumed to be constant. We use $X = 0.1$ in Figure 1 and $\rho = 0.05$ in Figure 2. The other parameter values are $T = 30, \lambda = 0.3$ and $m_0 = 0.08$. Here, the curve $r^*(t)$ separates the non-prepayment region in the above and the feasible prepayment region below. Since L is decreasing with respect to both ρ and r, we obtain a lower value of $r^*(t)$ when ρ assumes a higher value. The monotonicity properties of $r^*(t)$ with respect to ρ and X as deduced from Lemmas 1 and 2 are verified. We also observe that there exists some critical value of time to expiry such that the mortgage is never prepaid when the time to expiry is less than this critical value.

3 Second order accurate numerical schemes for solving penalty formulation

In this section, we propose two second order time accurate finite difference schemes for solving the penalty formulation presented in eq. (2.4). Goncharov (2004) analyzes the numerical procedure proposed by Stanton (1995) for solving eq. (2.4). He interprets Stanton's scheme as a first order time accurate fractional step method. Goncharov also cautions that numerical methods in general cannot be of second order accurate unless the discontinuity in the prepayment function is specially treated. Though Forsyth and Vetzal have employed the second order Crank-Nicolson discretization in their numerical scheme (2002), erratic convergence behaviors are observed in their American option calculations. In the implementation of their numerical method, a system of non-linear equations have to be solved in each time step due to implicit discretization of the non-linear penalty term.

We consider the construction of finite difference schemes for the following prototype equation (Forsyth and Vetzal, 2002), which is obtained from the penalty approximation of the linear complementarity formulation of the pricing model of an American option with payoff function $\phi(S)$:

$$\frac{\partial U}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} - rU + \rho \max(\phi(S) - U, 0). \tag{3.1}$$

Here, the option value function $U(S, \tau)$ is a function of stock price S and time to expiry τ, where $\tau = T - t$, and σ is the volatility of the stock price. In this section, we propose two modifications of the discretization of the penalty term in the Crank-Nicolson scheme proposed by Forsyth and Vetzal (2002). The objectives of which are to achieve better convergence behaviors and computational efficiency.

Let U_j^n denote the discrete numerical approximation to $U(S_j, \tau_n)$, where $S_j = S_0 + j\Delta S, j = 1, 2, \cdots, N_S$, and $\tau_n = n\Delta\tau, n = 1, 2, \cdots, N_\tau$. Here, ΔS and $\Delta\tau$ are the stepwidth and time step, respectively, N_S and N_τ are the total number of spatial and temporal grids, respectively. Define the spatial difference operator $\mathcal{L}_h$ by

$$\mathcal{L}_h U_j^n \quad = \quad \frac{\sigma^2}{2} S_j^2 \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta S^2} + \ rS_j \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta S} - rU_j^n.$$

Forsyth and Vetzal (2002) propose the following Crank-Nicolson discretizaiton of eq. (3.1):

$$U_j^{n+1} = U_j^n + \frac{\Delta\tau}{2} \frac{\mathcal{L}_h U_j^{n+1} + \mathcal{L}_h U_j^n}{2} + \xi^{n+1}[\phi(S) - U_j^{n+1}] \tag{3.2a}$$

where

$$\xi^{n+1} = \begin{cases} \rho\Delta\tau & \text{if } \phi(S) > U_j^{n+1} \\ 0 & \text{otherwise} \end{cases} . \tag{3.2b}$$

Since the penalty term $\xi^{n+1}[\phi(S) - U_j^{n+1}]$ is non-linear, one has to solve a non-linear system of algebraic equations at each time step to obtain U_j^{n+1}. The generalized Newton iteration procedure is adopted by Forsyth and Vetzal for the numerical solution of the non-linear system.

Experimental tests on the Forsyth-Vetzal scheme reveal that the numerical results may not exhibit quadratic rate of convergence when the Rannacher smoothing procedure is not applied. With some slight modification of the discretized penalty term in eqs. (3.2a,b), we may achieve better convergence behaviors and computational efficiency. Our proposed modifications are presented below.

Scheme One
The penalty term is discretized at $\left(n+\frac{1}{2}\right)^{\text{th}}$ time level, so that it now becomes

$$\xi^{n+\frac{1}{2}}\left[\phi(S)-\frac{U_j^{n+1}+U_j^n}{2}\right] \tag{3.3a}$$

where

$$\xi^{n+\frac{1}{2}}=\begin{cases}\rho\Delta\tau & \text{if } \phi(S)>\frac{U_j^{n+1}+U_j^n}{2}\\ 0 & \text{otherwise}\end{cases}. \tag{3.3b}$$

Scheme Two
The non-linearity in the penalty term disappears when we replace the implicit term $(U_j^{n+1}+U_j^n)/2$ by the explicit term $(3U_j^n-U_j^{n-1})/2$. Now, the solution for U_j^{n+1} amounts to the solution of a linear tridiagonal system of algebraic equations. The discretized penalty term is given by

$$\widehat{\xi}^{n+\frac{1}{2}}\left[\phi(S)-\frac{3U_j^n-U_j^{n-1}}{2}\right], \tag{3.4a}$$

where

$$\widehat{\xi}^{n+\frac{1}{2}}=\begin{cases}\rho\Delta\tau & \text{if } \phi(S)>\frac{3U_j^n-U_j^{n-1}}{2}\\ 0 & \text{otherwise}\end{cases}. \tag{3.4b}$$

Unfortunately, the new scheme is a three-level scheme. As part of the initiation procedure, one has to use an alternative two-level second order accurate scheme to obtain the numerical solution at the first time level.

To compare the convergence behaviors of the numerical results obtained using our Scheme One and the Forsyth-Vetzal scheme (without applying Rannacher smoothing), we repeated similar pricing calcaulations on an American put option as reported in Forsyth and Vetzal's paper (2002). The payoff of the put option is $\phi(S)=\max(X-S,0)$, where X is the strike price. The numerical results are presented in Table 1. The parameter values used in the American put option are: $r=0.1, S=100, \tau=0.25, X=100$. For a second order time accurate scheme, we expect that the error of approximation is reduced by a factor of $1/4$ when the number of time steps is doubled. The numerical solutions obtained from our Scheme One do demonstrate quadratic rate of convergence while those from the Forsyth-Vetzal scheme reveal erratic convergence behaviors.

We have also applied both our proposed schemes and Stanton's scheme to numerical valuation of a mortgage loan. In particular, we measured the mean squared errors and examined the rates of decrease of the mean squared errors with increasing number of time steps. In Table 2, we present the numerical results in a typical mortgage loan valuation. From the values of the successive ratios of the mean squared errors, we deduce that both Scheme One and Scheme Two are second order time accurate while Stanton's Scheme is only first order time accurate (since the error of approximation is halved when the number of time steps is doubled).

4 American warrant subject to issuer's calling

In this section, we consider the impact of issuer's calling right on the optimal early exercise policy of an American warrant. Examples of callable American warrants that are traded in the financial markets can be found in Kwok and Wu's paper (2000). The payoff of the American warrant upon exercise by the holder is the usual call option payoff: $\max(S_{\widehat{t}} - X, 0)$, where $S_{\widehat{t}}$ is the stock price at the exercise time $\widehat{t}$ and X is the strike price. Upon calling by the issuer, the American warrant is terminated prematurely. For simplicity in our subsequent analysis, we assume that the holder receives the fix dollar amount K as rebate from the issuer upon calling. The calling may be visualized as an event of pre-mature termination. In this sense, the callable American warrant can be interpreted as an American option subject to event risk of early termination (Szimayer, 2005). As noted by Szimayer, the presence of calling risk influences the optimal exercise policy adopted by the warrant holder.

We would like to derive the formulation of the pricing model of an American warrant subject to the risk of pre-mature termination by issuer's calling. Let $V(S,t)$ be the price function of the American warrant, where S is the stock price. Let $\phi(S)$ be payoff upon exercise and $\psi(S)$ be the rebate received by the holder upon calling. Assume the arrival of calling by issuer to be governed by a Poisson process with intensity $\rho \mathbf{1}_{\{V>\psi\}}$, where ρ is a constant. The indicator function $\mathbf{1}_{\{V>\psi\}}$ is included since the issuer calls only when $V > \psi$. Here, we consider a callable American warrant whose exercise payoff $\phi(S) = S - X$ and rebate $\psi(S) = K$.

Partial differential equation formulation
Under the risk neutral measure Q, the dynamic of the stock price is assumed to be governed by

$$\frac{dS}{S} = r\,dt + \sigma\,dZ, \tag{4.1}$$

assuming zero dividend yield of the stock. By following a similar argument as that of the value of the liability mortgage loan in Sec. 2, we obtain the following governing equation for $V(S,t)$ in the continuation region where the warrant remains alive. We obtain

$$rV\,dt = \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S}\right)dt + \rho\max(V-\psi,0)$$

so that

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + \rho\max(V-\psi,0). \tag{4.2}$$

In the stopping region where it is optimal for the holder to exercise the warrant, we have

$$V(S,t) = \phi(S). \tag{4.3}$$

We write $\tau = T-t$, where τ is the time to expiry and let $\mathcal{L}_w$ be the differential operator

$$\mathcal{L}_w = \frac{\sigma^2}{2}S^2\frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} - r,$$

the complementarity formulation for $V(S,\tau)$ is given by

$$\min\left(\frac{\partial V}{\partial \tau} - \mathcal{L}_w V + \rho\max(V-\psi,0), V-\phi\right) = 0$$

subject to terminal condition

$$V(S,0) = \max(\phi(S),0). \tag{4.4}$$

The above complementarity formulation is non-linear due to the presence of the term $\max(V-\psi,0)$. Recall that in the Carr-Linesky model of executive stock valuation, the intensity of the process of event arrival does not depend on the value function, so their differential equation formulation remains linear.

In the limit $\rho \to \infty$, the issuer calls whenever V reaches ψ from below. This corresponds to the "no delayed call" scenario. The complementarity formulation (4.4) becomes (Dai and Kwok, 2005)

$$\min\left(\max\left(\frac{\partial V}{\partial \tau} - \mathcal{L}_w V, V - \psi\right), V - \phi\right) = 0$$
$$V(S, 0) = \min(\phi, \psi). \tag{4.5}$$

Now, the pricing model becomes the *double obstacle* problem since V is bounded above by ψ and below by ϕ. The interaction of the optimal calling and exercise policies of the American warrant under optimality of calling have been discussed by Dai and Kwok (2005).

In the simple case of $\psi = K$ and $\phi = S - X$, the optimal calling and exercise policies are given by (Kwok and Wu, 2000)

$$S^*(\tau) = \min(\widetilde{S}^*(\tau), K + X), \tag{4.6}$$

where $S^*(\tau)$ denote the critical stock price at which the warrant is either optimally called by the issuer or exercised by the holder. Here, $\widetilde{S}^*(\tau)$ denotes the critical stock price of the usual American call option. Kwok and Wu show that there is a critical value τ^* of the time to expiry at which

$$\widetilde{S}^*(\tau) = K + X, \tag{4.7a}$$

and

$$S^*(\tau) = \begin{cases} \widehat{S}^*(\tau) & \text{when } \tau \leq \tau^* \\ K + X & \text{when } \tau > \tau^* \end{cases}. \tag{4.7b}$$

Under the present framework of modeling the arrival of calling as a Poisson process, we would like to examine the critical stock price $S^*_\rho(\tau)$ at which it is optimal for the holder to exercise. In Figure 3, we plot $S^*_\rho(\tau)$ against τ for varying values of ρ. The parameter values used in the calculations are: $X = 100, r = 0.02, q = 0.04, \sigma = 0.3, T = 2$ and $K = 130$. Interestingly, it is observed from the figure that

$$S^*_\rho(\tau) = \widetilde{S}^*(\tau) \quad \text{for} \quad \tau \leq \tau^* \tag{4.8}$$

and $S^*_\rho(\tau)$ is a decreasing function of ρ when $\tau > \tau^*$.

The monotonically decreasing property of $S^*_\rho(\tau)$ with respect to ρ can be inferred easily from the monotonicity of the value function $V(S, \tau; \rho)$ with respect to ρ. In particular, we have $V(S,\tau;\rho) \geq V(S,\tau;\infty)$ for any finite value of ρ. Recall that when $\tau \leq \tau^*$, the issuer would never call the warrant since

$$V(S,\tau;\rho) \leq C_A(S,\tau) \leq K, \quad \tau \leq \tau^*,$$

where $C_A(S,\tau)$ is the price function of the non-callable American call. For $\tau \leq \tau^*$, the callable American warrant is equivalent to its non-callable counterpart, hence the result in eq. (4.8).

5 Conclusion

The prepayment right in a mortgage loan or the calling right in a callable security represents pre-mature termination clause that limit the market value of liabilities of the financial security. When optimality of exercising the right is assumed, the pricing model of the derivative constitutes an optimal stopping problem. The optimal stopping rule and the price of the derivative can be obtained via the solution of the linear complementarity formulation of the optimal stopping problem. However, empirical studies have shown that these embedded rights are generally not be executed optimally by the mortgagor and security issuer. Suppose the propensity of mortgagor's prepayment or issuer's calling is modeled by the intensity of a Poisson process, we have shown that the pricing formulation resembles the penalty approximation approach for solving the linear complementarity formulation of an optimal stopping problem. The penalty parameter is seen to have a vivid financial interpretation. It can be visualized as the intensity of the Poisson process modeling the arrival of mortgagor's prepayment or issuer's calling.

For the mortgage loan valuation problem, the pricing of the value of liabilities remains a non-linear problem. The boundary that separates the non-prepayment region and feasible prepayment region is obtained as part of the solution of the pricing model. We have obtained several theoretical results on the monotonicity properties of the separating boundary with respect to the intensity of prepayment and level of transaction cost.

The pricing model of a callable American warrant can be formulated as a set of variational inequalities with double obstacle functions. This is because the value function is bounded above by the payoff upon issuer's calling and below by the payoff upon holder's exercise. When the arrival of issuer's calling is modeled by a Poisson process, the pricing model of the callable American warrant resembles that of an American call option subject to event risk of pre-mature termination. The double obstacle problem is simplified to a set of variational inequalities with one obstacle, except that one of the inequalities becomes non-linear. We have managed to deduce the optimal holder's exercise policy subject to risk of calling by issuer. There exists a critical value of time to expiry below which the warrant is always in the non-calling region. Beyond this critical time of expiry, the arrival of issuer's calling is plausible. We have shown that the critical stock price at which the holder should exercise decreases with increasing value of intensity of calling by the issuer.

We have proposed two versions of second order finite difference schemes for solving the penalty approximation of the linear complementarity formulation of an optimal stopping problem. With proper explicit discretization of the non-linear penalty term, we achieve better computational efficiency and convergence behaviors when compared to other existing schemes in the literature.

References

[1] Khaliq, A.Q.M., D.A. Voss and S.H.K. Kazmi, 2005, "Adaptive θ-methods for pricing American options", Working paper of Middle Tennessee State University.

[2] Carr, P. and V. Linetsky, 2000, "The valuation of executive stock options in an intensity-based framework," *European Finance Review* 4, 211-230.

[3] Dai, M. and Y.K. Kwok, 2005, "Optimal policies of call with notice period requirement," Working paper of National University of Singapore.

[4] Deng, Y., J.M. Quigley, and R. Van Order, 2000, "Mortgage terminations, heterogeneity and the exercise of mortgage options," *Econometrica* 68, 275-307.

[5] d'Halluin, Y., P.A. Forsyth and G. Labahn, 2004, "A penalty method for American options with jump diffusion processes", *Numerische Mathematik* 97, 321-352.

[6] Forsyth, P.A. and K.R. Vetzal, 2002, "Quadratic convergence for valuing American options using a penalty method, "*SIAM Journal on Scientific Computation* 23, 2096-2123.

[7] Friedman, A., 1982, *Variational principles and free boundary problems*, John Wiley and Sons, Inc.

[8] Goncharov, Y., 2004, "An intensity-based approach to valuation of mortgage contracts subject to prepayment risk," Working paper of University of Michigan.

[9] Kwok, Y.K. and L. Wu, 2000, "Effects of callable feature on early exercise feature," *Review of Derivatives Research* 4, 189-211.

[10] Lau, K.W. and Y.K. Kwok, 2004, "Anatomy of option features in convertible bonds," *Journal of Futures Markets* 24, 513-532.

[11] McConnell, J.J., and M. Singh, 1994, "Rational prepayments and the valuation of collateralized mortgage obligations," *Journal of Finance* 49, 891-921.

[12] Schwartz, E.S., and W.N. Torous, 1992, "Prepayment, default, and the valuation of mortgage pass-through securities," *Journal of Business* 65, 221-239.

[13] Stanton, R., 1995, "Rational prepayment and the valuation of mortgage backed securities," *Review of Financial Studies* 8, 677-708.

[14] Szimayer, A., 2004, "A reduced form model for ESO valuation," *Mathematical Methods of Operations Research* 59(1), 111-128.

[15] Szimayer, A., 2005, "Valuation of American options in the presence of event risk," *Finance and Stochastics* 9, 89-107.

[16] Zvan, R., P.A. Forsyth and K.R. Vetzal, 1998, "Penalty methods for American options with stochastic volatility," *Journal of Computational and Applied Mathematics* 91, 199-218.

Appendix A – Proof of Lemma 1

Assume the contrary, that is, $L_1 > L_2$ on an open set G so that

$$\max(L_1 - \psi, 0) \geq \max(L_2 - \psi, 0).$$

Consequently, we have

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \mathcal{L}\right)(L_1 - L_2) \\
= \;& \rho_1 \max(L_1 - \psi, 0) - \rho_2 \max(L_2 - \psi, 0) + f_2 - f_1 \\
= \;& \rho_1 \left[\max(L_1 - \psi, 0) - \max(L_2 - \psi, 0)\right] + (\rho_1 - \rho_2) \max(L_2 - \psi, 0) \\
& + (f_2 - f_1) \geq 0.
\end{aligned}$$

On the boundary of G, we have $L_1 = L_2 = 0$ on ∂G. By using the comparison principle of the linear operator $-\left(\frac{\partial}{\partial t} + \mathcal{L}\right)$, we obtain $L_1 \leq L_2$. A contradiction is encountered, so we have the desired result.

Appendix B – Proof of Lemma 2

It is reasonable to assume $-\frac{dP(t)}{dt} + rP(t) \geq 0$ since the mortgage loan payment is amortized throughout the life of the contract. For $X_1 < X_2$, it suffices to show

$$L(r, t; X_1) - P(t)(1 + X_1) \geq L(r, t; X_2) - P(t)(1 + X_2)$$

or equivalently

$$L(r, t; X_1) \geq L(r, t; X_2) - P(t)(X_2 - X_1).$$

We define

$$\widetilde{L}(r, t) = L(r, t; X_2) - P(t)(X_2 - X_1).$$

It is easy to check that

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\mathcal{L}\right)\widetilde{L} &= \left(\frac{\partial}{\partial t}\mathcal{L}\right)L(r,t;X_2) \\ &\quad + (X_2-X_1)\left[-\frac{dP(t)}{dt}+rP(t)\right].\end{aligned}$$

Note that

$$(X_2-X_1)\left(-\frac{dP(t)}{dt}+rP(t)\right)\geq 0.$$

By Lemma 1, we infer that $\widetilde{L}(r,t)\leq L(r,t;X_1)$, hence the desired result.

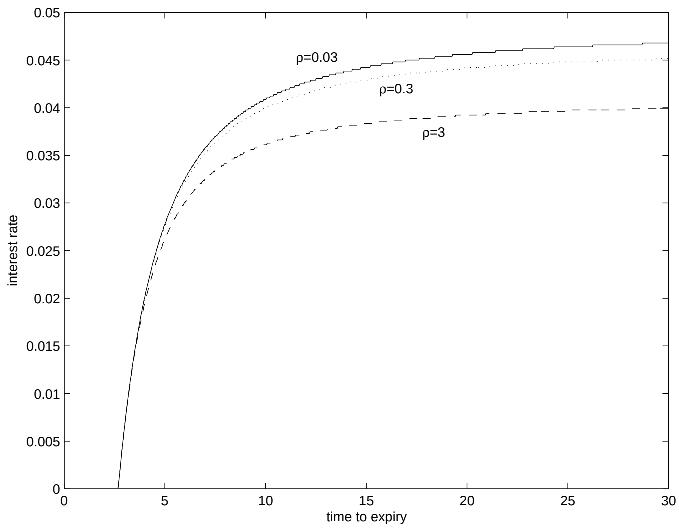

Figure 1 Plot of the separating boundary r^* against time to expiry $T - t$ with varying values of the penalty parameter ρ. The non-prepayment region lies above the separating boundary curve. With a higher value of $\rho, r^*(t)$ assumes a lower value.

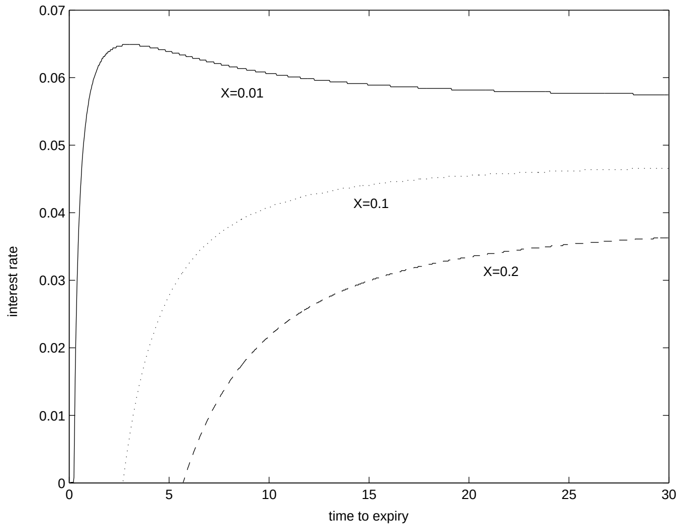

Figure 2 Plot of the separating boundary r^* against time to expiry $T - t$ with varying values of the transaction cost factor X. With a higher value of X, r^* assumes a lower value.

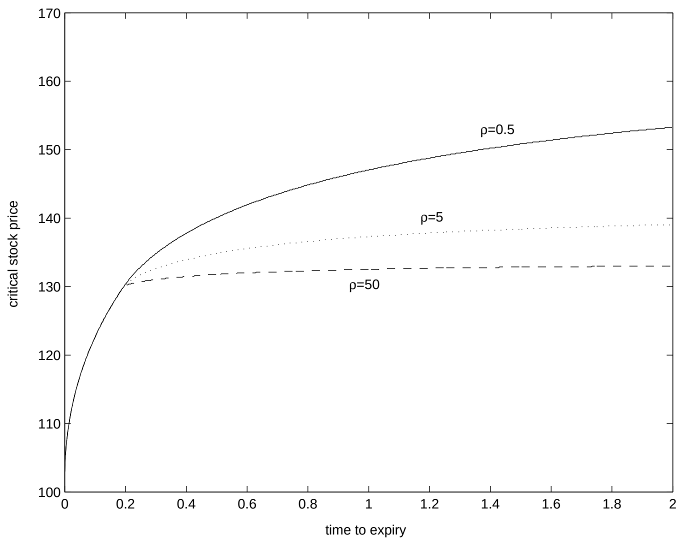

Figure 3 Plot of the critical stock price S^* against time to expiry $T - t$ with varying values of the penalty parameter ρ. With a higher value of ρ, S^* assumes a lower value.

		Scheme One			Forsyth-Vetzal's scheme		
N_τ	N_S	Value	Change	Ratio	Value	Change	Ratio
$\sigma = 0.2$, spatial domain of computation $= [0, 200]$							
100	200	3.06908			3.06751		
200	400	3.06986	0.00078		3.06910	0.00159	
400	800	3.07005	0.00019	4.1	3.06968	0.00058	2.8
800	1600	3.07010	0.00005	4.1	3.06991	0.00023	2.5
1600	3200	3.07010	0.00001	4.3	3.07001	0.00010	2.3
$\sigma = 0.8$, spatial domain of computation $= [0, 1000]$							
100	200	14.67638			14.67431		
200	400	14.67826	0.00188		14.67729	0.00298	
400	800	14.67873	0.00047	4.0	14.67828	0.00099	3.0
800	1600	14.67885	0.00012	4.0	14.67863	0.00035	2.8
1600	3200	14.67887	0.00002	4.2	14.67877	0.00014	2.5

Table 1 Comparison of the convergence behaviors of pricing calculations of an American put option using Scheme One and Forsyth-Vetzal's scheme (without applying Rannacher smoothing). The numerical results obtained from Scheme One demonstrate quadratic rate of convergence.

		Stanton's Scheme		Scheme One		Scheme Two	
N_τ	N_S	mean squared errors	ratio	mean squared errors	ratio	mean squared errors	ratio
200	200	13.468	—	0.062315	—	0.038450	—
400	400	6.706	2.0	0.011925	5.2	0.009535	4.0
800	800	3.347	2.0	0.002557	4.7	0.002374	4.0
1600	1600	1.672	2.0	0.000596	4.3	0.000591	4.0
3200	3200	0.835	2.0	0.00146	4.1	0.000147	4.0

Table 2 Examination of the rate of convergence of numerical calculations of pricing a mortgage loan using Stanton's scheme and our proposed modified Crank-Nicolson schemes.